Mathematics Spectrum

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REGULARS
CONCEPT MAP
SHORTCUT METHODS
From The Editor's Desk
Exam Tactics: Tacking JEE Made Easy

JEE
Final Touch
Fast Track Revision

STRAIGHT LINE
CONIC SECTION
MATHEMATICAL REASONING
FROM CLASS 11 SYLLABUS

APPLICATION OF INTEGRALS
DIFFERENTIAL EQUATION
MATRICES AND DETERMINATION
FROM CLASS 12 SYLLABUS

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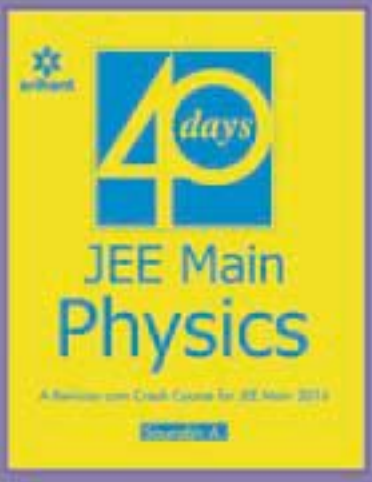

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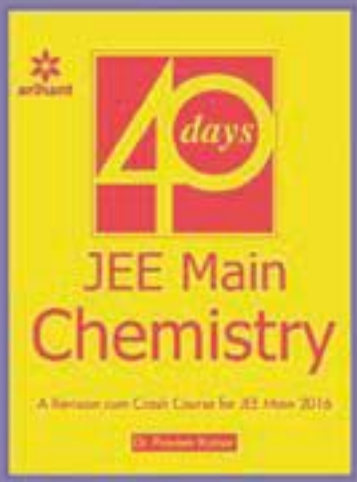

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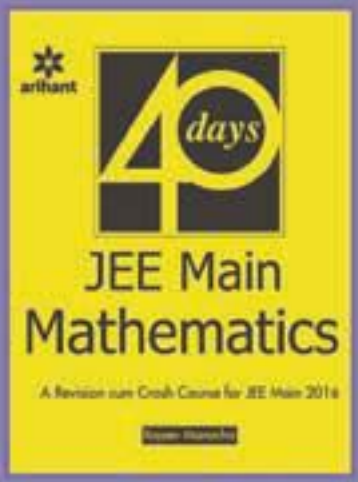

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CONTENTS

02 FROM THE EDITOR'S DESK
EXAM TACTICS : TACKLING JEE MADE EASY

04 **JEE FINAL TOUCH**
Fast Track Revision @ Class 11th Syllabus

46 **TEST TUNER**
JEE Main Scale Up

48 **CONCEPT MAP**
Binomial Theorem

50 **JEE FINAL TOUCH**
Fast Track Revision @ Class 12th Syllabus

76 **TEST TUNER**
JEE Main Scale Up

78 **TEST TUNER**
JEE Advanced Scale Up

82 **SHORTCUT METHODS**

84 **SPECTRUM FOR YOU**

87 **TEST ZOOM**
Board Exam Scale Up

89 **TEST ZOOM**
Board Exam Scale Up

95 **PARTICIPATE & WIN**

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EXAM TACTICS: Tackling JEE Made Easy

The present structure of JEE includes a two tier examination system i.e. JEE Mains and JEE Advanced. Let us explore them one by one

JEE MAINS PAPER

In Mathematics, it is of common observation for JEE Mains that the subject papers of Physics and Chemistry are usually simple and comparatively easier, but the Mathematics paper is traditionally tougher. Most of the problems of this segment involve a lot of calculation and are time consuming. Thus, among all the three sections, this particular section is the most difficult one. In present scenario, the high difficulty level of the question paper is mainly due to its deviation from NCERT textbooks. The approximate weightage of different topics in the question paper is shown below:

Algebra	10-11 questions
Trigonometry	2-5 questions
Calculus	5-8 questions
2 D coordinate Geometry	5-7 questions
Vector Algebra (3 D Geometry)	3-5 questions
Mathematical Reasoning	1-2 questions
Statistics	1-2 questions
Miscellaneous	1-2 questions

The analysis of Mathematics paper of JEE Mains 2015 suggests 54% participation of class 11th syllabus and rest 47% of class 12th syllabus. Difficulty level wise also this sort of difference is noticed, i.e. the percentage of easy questions was less (36%), the percentage of tough questions was more or less same (16%), but the percentage of mediocre questions was substantially high (48%).

The last 2 year analysis of JEE Mains paper also suggests maximum patch up of question paper with NCERT in Chemistry (approximately 90%) and minimum in the case of Mathematics (30-45%).

MANTRAS TO CRACK JEE MAINS

The two tier examination pattern was introduced in 2013. The weightage of board percentage in final merit was one of the features of this system. The JEE Mains paper of this examination system is comparable with the IIT JEE screening examination of late nineties and early twenties.

For successfully facing this kind of examination try to focus on following points:

- One must prepare for final exam (JEE Advanced) only, i.e. do not target Mains examinations first. In my opinion **"Chote exams target karne se kabhi bhi bade results nahin milte"**, i.e. targeting smaller exams never gives big results. If you consider JEE Advanced as your target then Mains will be cleared automatically. In my opinion Mains is only a screening to differentiate between serious and non-serious aspirants. Thus, if you are among the serious lot then Mains will be of no problem to you.
- The analysis of 2013, 2014 and 2015 JEE Mains paper clearly suggests the importance of class 11th syllabus for this paper, i.e. 46% in Physics, 33% in Chemistry and 57% in Mathematics. So, a real grip over class 11th syllabus is required to succeed in JEE Mains paper.
- Nobody can deny the importance of NCERT Textbooks for this examination. Thus, a good command over NCERT textbooks is required for a great performance in this examination.
- For capturing NCERT line by line very recently, I saw Master the NCERT. A book, in which objective type questions are framed on each and every line of NCERT. This book can be very helpful to get success in this examination.
- Choose offline or online paper in accordance of your comfort zone and fine preparation. Both the examinations have the own advantages or disadvantages. The difficulty levels are also more or less same in both the formats of JEE Mains but on analysis it is quite visible that the online paper is framed on a slightly tougher side.
- Although, the questions paper contains all the three subjects and merit formation includes total marks obtained in the paper but remember subject wise cut off also exists in JEE. So, a balance in preparation is required subject wise to save you from such a subject wise cut off.

  As all the questions are single answer type MCQs hence, use **elimination technique** in those questions where you don't have clue about the correct option, rather you have clue about incorrect options. Eliminate 3 incorrect options to get the most probable correct option.
- As negative marking is there, hence be cautious in answering the questions. Do not take so much of risks.

JEE ADVANCED PAPER

The JEE Advanced paper is based upon old IIT JEE paper till 2013. It has 2 papers in total, i.e. first in the morning session (9 am to 12 pm) and second in the afternoon session (2 pm-5 pm). Both the papers contain 20 questions per subject, i.e. total 60 questions per paper. The question paper pattern is not at all fixed.

In 2015 Advanced paper following pattern was followed:

PAPER 1

- Section 1 with 8 Integer Type Questions
- Section 2 with 10 MCQs having One or More than One Correct Option
- Section 3 with 2 Matrix Match Questions.

PAPER 2

- Section 1 with 8 Integer Type Questions
- Section 2 with 8 MCQs having One or More than One Correct Option
- Section 3 with 2 Paragraphs having 2 questions each.

In both the papers, section 2 and 3 had a negative marking of 50%.

However, in 2014 JEE Advanced questions paper following pattern was followed:

PAPER 1

- 10 Multiple Choice Questions which are one or more than one answer correct.
- 10 Integer Type Questions. All the 20 questions with 3 marks each with no negative marking.

PAPER 2

- 10 Multiple Choice Questions with only one answer correct.
- 3 paragraphs with 2 question each having only One Answer Correct.
- A Matching Type MCQs with one answer correct.
- All the questions with 3 marks each and there is the negative marking of 1/3, i.e. +3 for correct answer and –1 for wrong answer.

Thus, the uniqueness of JEE Advanced paper lies in 2 facts i.e.

Its surprising element Every year JEE Advanced paper incorporates one or more surprise elements, e.g. in 2014, negative marking in paper 2 was the surprised element. Similarly, the frame of question paper was changed in 2013 as compared to the one seen in 2012 IIT-JEE paper.

Its variety of questions In both 2015 and 2014 JEE Advanced paper following type of questions were asked

- Single answer correct (MCQs)
- More than one answer correct (MCQs)
- Integer type question
- Linked comprehension type
- Matching type

The matrix match type which was the usual component of IIT JEE papers before 2013, become reintroduced in 2015. Likewise comprehension linked questions were 2 per passage in 2015 and 2014, while it was 3 per passage in IIT JEE before 2013.

So, we can say the nature of JEE Advanced paper is very dynamic. No pattern is fixed for the paper. The syllabus wise distribution of marks is more or less same as that seen in JEE Mains paper.

MANTRAS TO CRACK JEE ADVANCED

I personally feel that JEE Advanced is one of the toughest exam, among those of same nature we have worldwide. I also believe in the fact that *more organised examinations are always easier to crack*. So, although it is toughest pattern or frame wise, but if we planned scientifically than we can hit the bull's eye.

For scientific planning following points must be kept in mind:

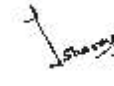
- Phase wise learning and practice is a must to crack these kinds of examinations. After having the analysis of previous year exams some conclusions can be drawn easily like, those about?
  - (i) the weightage of class 11th syllabus.
  - (ii) the difficulty level distribution among the questions of previous year papers.
  - (iii) the topic wise, question wise distribution of questions
  - (iv) the topic wise, marks wise distribution

  Such conclusions will help in strategy development according to your strength and weaknesses.
- If you will be approaching phase wise towards this examination, then it is very clear that till Mains you would be having very less opportunities to practice different question types which are the features of JEE Advanced only. Utilise the period between JEE Mains and JEE Advanced for this purpose. A mass of students waste their some of this precious time in the wait of Mains result. Don't ever commit this kind of mistake. Be optimistic, if you have JEE advanced as the target from day 1 in your mind than, you would clear JEE Mains with surety.
- The surprise attack is one of the most lethal weapon of this examination, hence, those aspirants who don't have any kind of premature frame about the pattern of question paper in their mind, has better chances of hitting the target. So, never create the premature frame of question paper in your mind.
- The large time span (9 am to 5 pm) is a very crucial factor in this examination. Such a large time span creates tiredness of all kind (i.e. physical as well as mental) in any person. So during the gap of 2 hrs between Paper 1 and 2 try to relax as much as you can. *Remember! The more you relax, better you perform.*

Last but not the least, a balance between the approaches towards Subjective and Objective formats is required to succeed in IIT as a whole as in present format the final merit also includes the weightage of your Board examinations.

Always keep in mind the words of Sir Winston Churchill

"The pessimist sees difficulty in every opportunity.
The optimist sees opportunity in every difficulty."



@CLASS XI SYLLABUS

JEE Final Touch

Fast Track Revision

Straight Lines

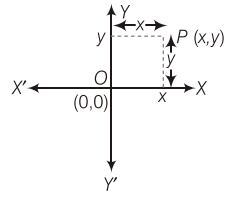
A straight line is a curve such that the line segment joining any two points on it lies wholly on it. It is one of the important topic which is frequently asked in JEE Main and Advanced. For sake of convenience, we are dividing this topic into seven sub-topics, *viz.* 'Coordinate systems', 'Locus of a point', 'Equations of straight line in different forms', 'Family of lines and length of perpendicular ', 'Position of points relative to a line and equation of angle bisectors', 'Pair of straight lines' and 'Reflection'.

COORDINATE SYSTEMS

There are various types of coordinate systems to represent a point in two dimensions, but here we will only discuss cartesian coordinate system and polar coordinate system.

Cartesian Coordinate System

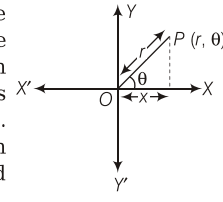
If we draw two mutually perpendicular lines in a plane, then they divide the plane in four parts, each part is known as I, II, III and IV quadrants and their point of intersection is called origin. Horizontal and vertical lines are known as X and Y-axes, respectively. Every point in the plane is located in the form of ordered pair (x, y), where x denotes the distance of the point from Y-axis and y is distance from X-axis. If a point lies on X-axis, then it is in the form of (x, 0) and on Y-axis, it is (0, y).



Since, $X'OX$ is perpendicular to $Y'OY$, this system of representation is also called rectangular coordinate system or orthogonal coordinate system.

Polar Coordinate System

Another way to locate the point in the plane is polar form (r, θ). Here, r is the distance of the point from the origin and θ is angle of inclination of radius vector (**OP**) with the positive X-axis. If coordinates of the point P in cartesian and polar form are (x, y) and (r, θ) respectively, then



$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\therefore \quad \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Some Basic Results

(i) **Distance formula** If two points are $A(x_1, y_1)$ and $B(x_2, y_2)$, then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and if $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ are polar coordinates, then $AB^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

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- Distance between $(x_1, 0)$ and $(x_2, 0)$ is $|x_1 - x_2|$.
- Distance between $(0, y_1)$ and $(0, y_2)$ is $|y_1 - y_2|$.

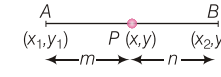
(ii) **Section formulae** If two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given and $P(x, y)$ is any point, which divides the line segment AB in the ratio of $m : n$, then we can find the coordinates of $P(x, y)$.

(a) If P lies in between A and B (internally), then

$$x = \frac{mx_2 + nx_1}{m + n}$$

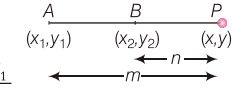
and

$$y = \frac{my_2 + ny_1}{m + n}$$



(b) If P lies on AB produced (externally), then

$$x = \frac{mx_2 - nx_1}{m - n} \text{ and } y = \frac{my_2 - ny_1}{m - n}$$



Note If (x_1, y_1) and (x_2, y_2) are coordinates of points A and B respectively, then the coordinates of mid-point of line segment joining A and B are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

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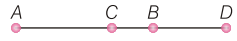
- To find ratio, consider P divides the given line segment in $k:1$, then coordinates of P are given by $\left(\frac{x_1+kx_2}{k+1}, \frac{y_1+ky_2}{k+1}\right)$. If k is positive, then it divides the line segment internally and for negative, it divides the line segment externally.
- To prove that four points A, B, C and D are vertices of
  (a) **Parallelogram** show that diagonals AC and BD have same mid-point.
  (b) **Rhombus** show that diagonals have same mid-point and adjacent sides are equal.
  (c) **Square** show that diagonals are equal and adjacent sides are equal.
  (d) **Rectangle** show that diagonals are equal and they have same mid-point.

(iii) **Harmonic conjugates** Let A and B be two given points. Now, if the point C divides line segment AB internally in the ratio $m:n$ and the point D divides the line segment AB externally in the ratio $m:n$, then the points C and D are called the harmonic conjugates with respect to points A and B.



Note $|\overline{AC}|, |\overline{AB}|, |\overline{AD}|$ are in harmonic progression,
i.e. $\frac{2}{|\overline{AB}|} = \frac{1}{|\overline{AC}|} + \frac{1}{|\overline{AD}|}$

(iv) **Area of triangle** If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle, then area of triangle is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

or

$$\frac{1}{2}\left|[x_1\ (y_2 - y_3) + x_2\ (y_3 - y_1) + x_3\ (y_1 - y_2)]\right|$$

Note (i) If three points A, B and C are collinear, then area of ΔABC is zero and conversely, if area of ΔABC is zero , then three points A, B and C are collinear.
(ii) Triangle having integral coordinates can never be equilateral.

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- Area of triangle, having vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, can also be found by easy method, i.e. stair method which is given below

Area of $\Delta ABC = \frac{1}{2}\left|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{matrix}\right|$

$$= \frac{1}{2}\left|\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}\right|$$

- If $a_r x + b_r y + c_r = 0\ (r = 1, 2, 3)$ are the equations of sides of a triangle, then the area of the triangle is given by

$$\frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2, C_3 are the cofactor of c_1, c_2, c_3 in the determinant.
- Area of triangle formed by coordinate axes and the line

$$ax + by + c = 0 \text{ is } \frac{c^2}{2ab}$$

(v) **Area of polygon** If $(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots (x_n, y_n)$ are the vertices of a polygon, then area of polygon is given by

$$\frac{1}{2}\left\|\begin{matrix} x_1 & x_2 \\ y_1 & y_2 \end{matrix}\right| + \left|\begin{matrix} x_2 & x_3 \\ y_2 & y_3 \end{matrix}\right| + \left|\begin{matrix} x_3 & x_4 \\ y_3 & y_4 \end{matrix}\right| + \ldots + \left|\begin{matrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{matrix}\right| + \left|\begin{matrix} x_n & x_1 \\ y_n & y_1 \end{matrix}\right\|$$

or

$$\frac{1}{2}\left|\{(x_1\ y_2 - x_2y_1) + (x_2y_3 - y_2x_3) + (x_3y_4 - y_3\ x_4) + \ldots + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)\}\right|$$

SPEED BOOSTER

- Using stair method, we have

Area of polygon $= \frac{1}{2}\left|\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{matrix}\right| = \frac{1}{2}\left|\{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (x_2y_1 + x_3y_2 + \ldots + x_1y_n)\}\right|$

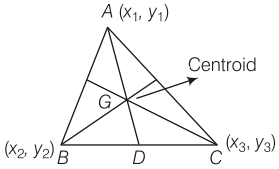
- If the two opposite vertex of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$, then its area

$$= \frac{1}{2}AC^2 = \frac{1}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

- Area of rhombus formed by $|ax| + |by| + |c| = 0$ is $\frac{2c^2}{ab}$.

Standard Points of a Triangle

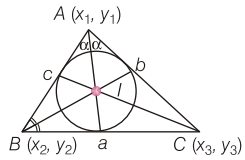
(i) **Centroid** The point of intersection of the medians of triangle is called the centroid of the triangle. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are vertices of the triangle, then coordinates of the centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.



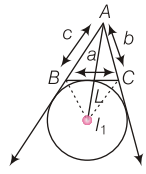
(ii) **Incentre** The point of intersection of internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre of a triangle with vertices as $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

where, $a = BC$, $b = AC$ and $c = AB$.



(iii) **Excentre** The point of intersection of external bisector of the two angles and internal angle bisector of third angle is called excentre. Hence, for a triangle three excentres are possible, opposite to each vertex.



(a) Excentre opposite to A

$$= \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

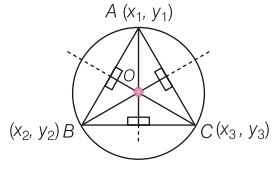
(b) Excentre opposite to B

$$= \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$

(c) Excentre opposite to C

$$= \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

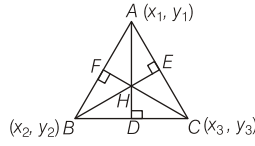
(iv) **Circumcentre** The point of intersection of perpendicular bisectors of the sides of a triangle is called circumcentre of triangle. This is the point which is at equidistant from the vertices of the triangle.



If O is circumcentre, then $AO = BO = CO$ will give two equations. By simplifying it, we can get coordinates of circumcentre. Also, we can get coordinates of circumcentre by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

(v) **Orthocentre** The point of intersection of the altitudes drawn from the vertices of the triangle is called orthocentre of the triangle.

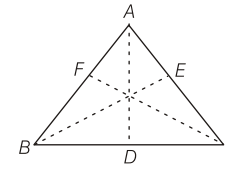


If H is orthocentre of the ΔABC, then coordinates of orthocentre are

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

SPEED BOOSTER

- If angle bisectors of ΔABC intersect the BC, AC and AB at D, E and F, respectively.
  Also, $|AB| = c, |BC| = a, |CA| = b$, then



  $AF = AE = s - a$, $BF = BD = s - b$
  and $CD = CE = s - c$
  where, s is semi-perimeter of ΔABC.
- Orthocentre of the triangle having vertices (a, b), (b, a) and (a, a) is (a, a).
- If the circumcentre and centroid of a triangle are respectively (a, b) and (c, d), then orthocentre is $(3c - 2a, 3d - 2b)$.

(vi) **Nine point centre** Let ABC be a triangle such that AD, BE and CF are its altitudes drawn from A, B and C on the opposite sides, H, I, J are the mid-points of the sides BC, CA and AB respectively and K, L, M are the mid-points of the line segments joining the orthocentre O' to the angular points A, B, C. These nine points (D, E, F, H, I, J, K, L, M) are concyclic and the circle passing through these nine points is called the nine point circle. Its centre is known as the nine point centre.

The nine point centre of a triangle is collinear with the circumcentre O and the orthocentre O' and bisects the segment joining them. The radius of nine point circle is half the radius of the circumcircle.

REMEMBER

- Centroid of the triangle formed by joining the middle points of the sides of a triangle is same as the centroid of the triangle.
- Centroid divides the median in 2 : 1 (2 from the vertex and 1 from the opposite side).
- Incentre divides the angle bisector in following ratios
  $b + c : a, c + a : b, a + b : c$
- Circumcentre, centroid and orthocentre of an acute triangle are collinear. Also, centroid divides the join of orthocentre and circumcentre in the ratio 2 : 1.
- In an equilateral triangle, centroid, orthocentre, incentre and circumcentre coincide. In an isosceles triangle, all these four points lie on the same line.
- In right angled triangle, orthocentre lies at the vertex which forms right angle.

Best Practice SHOTS

1. The line joining $A(b\cos\alpha, b\sin\alpha)$ and $B(a\cos\beta, a\sin\beta)$ is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then $x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right)$ is equal to

(a) -1 *(b)* 0 *(c)* 1 *(d)* $a^2 + b^2$

2. If the coordinates of two opposite vertices of a square are $(6, -3)$ and $(-2, 3)$, then area of square is

(a) 50 sq units *(b)* 25 sq units *(c)* 35 sq units *(d)* None

3. If a vertex of a triangle is $(1, 1)$ and the mid-points of two sides through this vertex are $(-1, 2)$ and $(3,2)$, then the centroid of the triangle is

(a) $\left(1, \frac{7}{3}\right)$ *(b)* $\left(\frac{1}{3}, \frac{7}{3}\right)$ *(c)* $\left(-1, \frac{7}{3}\right)$ *(d)* $\left(\frac{-1}{3}, \frac{7}{3}\right)$

4. The x-coordinate of the incentre of the triangle that has coordinates of mid-points of its sides as $(0,1)$, $(1, 1)$ and $(1, 0)$ is **[JEE Main 2013]**

(a) $2 + \sqrt{2}$ *(b)* $2 - \sqrt{2}$ *(c)* $1 + \sqrt{2}$ *(d)* $1 - \sqrt{2}$

5. The coordinates of the circumcentre of triangle whose vertices are $A(5, -1)$, $B(-1, 5)$ and $C(6, 6)$, are

(a) $\left(\frac{8}{23}, \frac{8}{23}\right)$ *(b)* $\left(\frac{23}{16}, \frac{23}{16}\right)$ *(c)* $\left(\frac{23}{8}, \frac{23}{8}\right)$ *(d)* None

6. The circumcentre of a triangle having vertices $A(a, a\tan\alpha)$, $B(b, b\tan\beta)$, $C(c, c\tan\gamma)$ is at the origin, where $\alpha + \beta + \gamma = \pi$. If orthocentre lies on the line $\left(4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)x - \lambda y = 0$, then λ is

(a) $4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

(b) $2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

(c) $1 + 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

(d) None of the above

7. If a vertex, the circumcentre and the centroid of a triangle are $(0,0)$, $(3,4)$ and $(6, 8)$ respectively, then the triangle must be

(a) a right angled triangle
(b) an equilateral triangle
(c) an isosceles triangle
(d) a right angled isosceles triangle

LOCUS OF A POINT

Locus

The path described by a point which moves under given geometrical condition or conditions is called its locus.

Some important loci are as follow

(i) Locus of a point, which is equidistant from a fixed point, is circle.

(ii) If sum of the distances of a point from two fixed points is always same, then locus is called ellipse. If A and B are two fixed points and P is any point such that $PA + PB = 2a$, then P will form ellipse.

(iii) If difference of the distances of a point from two fixed points is always same, then locus is called hyperbola. If A and B are two fixed points and P is any point such that $|PA - PB| = 2a$, then P will form hyperbola.

(iv) Locus of point at equidistant from two fixed points is called perpendicular bisector. If A and B are two fixed points and P is any point such that $PA = PB$, then the locus of P will be perpendicular bisector of line segment AB.

Process to Find the Locus of a Point

Step I Let (h, k) be the coordinates of the moving point say P.

Step II Apply the geometrical conditions on (h, k) and establish the relation between h and k.

Step III Now, replace h by x and k by y, then the obtained equation is required locus.

Transformation or Change of Axes

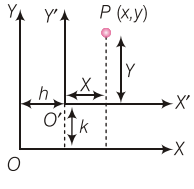
The coordinates of the points and equation of the curve changes when either the origin is changed or direction of axes or both are changed. This process is called transformation of axes or change of axes.

Change of Origin (Translation of Axes)

If (x, y) are coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to new axes, then $x = X + h$ and $y = Y + k$, where (h, k) are coordinates of shifted origin.

(a) The point whose coordinates are (x, y) has now the coordinates $(x - h, y - k)$.

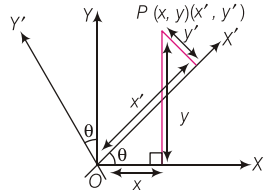
(b) When origin O is shifted to point $O'(h, k)$ without rotation of axes, then new equation of curve can be obtained by putting $(X + h)$ in place of x and $(Y + k)$ in place of y.



Rotation of Axes without Changing the Origin

Let OX, OY be the original axes and OX', OY' be the new axes obtained by rotating OX and OY through an angle θ in anti-clockwise sense.

Let P be any point in the plane having coordinates (x, y) with respect to OX and OY and (x ′, y ′) with respect to axes OX' and OY'.



Then, $x = x'\cos\theta - y'\sin\theta$; $y = x'\sin\theta + y'\cos\theta$

and $x' = x\cos\theta + y\sin\theta$; $y' = -x\sin\theta + y\cos\theta$

Relation between (x, y) and (x', y')

	$x\downarrow$	$y\downarrow$
$x'\rightarrow$	$\cos\theta$	$\sin\theta$
$y'\rightarrow$	$-\sin\theta$	$\cos\theta$

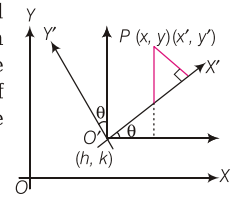
Note (i) The axes rotation in anti-clockwise direction is positive and clockwise rotation of axis is negative.

(ii) In order to remove the xy term from $f(x,y) = ax^2 + 2hxy + by^2$ without changing the origin, rotate the axes through an angle θ, where θ is given by $\tan 2\theta = \frac{2h}{a-b}$ or $\cot 2\theta = \frac{a-b}{2h}$ according as $a \neq b$ or $a = b$.

Change of Origin and Rotation of Axes

If origin O is changed to $O'(h, k)$ and axes are rotated about the new origin O' by an angle θ in the anti-clockwise sense such that new coordinates of $P(x, y)$ become (x', y'), then the equations of transformation will be

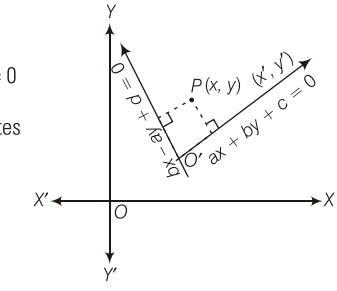
$$x = h + x'\cos\theta - y'\sin\theta \text{ and}$$
$$y = k + x'\sin\theta + y'\cos\theta$$



MATHS MAXIMA

If $P(x, y)$ and two mutually perpendicular lines $ax + by + c = 0$ and $bx - ay + d = 0$ are taken as new axes such that new coordinates of P are (x', y'), then

$$x' = \frac{bx - ay + d}{\sqrt{b^2 + a^2}} \text{ and}$$
$$y' = \frac{ax + by + c}{\sqrt{a^2 + b^2}}.$$



Best Practice SHOTS

8. If the point $P(x, y)$ is equidistant from the points $A(a + b, b - a)$ and $B(a - b, a + b)$, then

(a) $ax = by$
(b) $bx = ay$ and P can be (a, b)
(c) $x^2 - y^2 = 2(ax + by)$
(d) None of the above

9. Locus of the centroid of the triangle whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and $(1,0)$, where t is a parameter , is

(a) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(b) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
(c) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
(d) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$

10. Vertices of a triangle are $(3, 4)$, $(5\cos\theta, 5\sin\theta)$ and $(5\sin\theta, -5\cos\theta)$, where $\theta \in R$. Locus of its orthocentre is

(a) $(x + y - 1)^2 + (x - y - 7)^2 = 100$
(b) $(x + y - 7)^2 + (x - y - 1)^2 = 100$
(c) $(x + y - 7)^2 + (x + y - 1)^2 = 100$
(d) $(x + y - 7)^2 + (x - y + 1)^2 = 100$

11. If the coordinates of a point (4, 5) becomes (–3, 9), then the point of origin be shifted at

(a) (7, – 4) (b) (7, 4)
(c) (–7, 4) (d) None of these

12. Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant term. The suitable point is

(a) $\left(\frac{3}{4}, 2\right)$ (b) $\left(\frac{3}{4}, -2\right)$
(c) $\left(-\frac{3}{4}, -2\right)$ (d) $\left(-\frac{3}{4}, 2\right)$

13. If the axes be turned through an angle $\tan^{-1} 2$. What does the equation $4xy - 3x^2 = a^2$ become?

(a) $X^2 - 4Y^2 = a^2$ (b) $X^2 + 4Y^2 = a^2$
(c) $X^2 + 4Y^2 = -a^2$ (d) None of these

14. If the axes are rotated through an angle of 45° in the clockwise direction, the coordinates of a point in the new system are (0,–2). Then, its original coordinates are

(a) $(\sqrt{2}, \sqrt{2})$ (b) $(-\sqrt{2}, \sqrt{2})$
(c) $(\sqrt{2}, -\sqrt{2})$ (d) $(-\sqrt{2}, -\sqrt{2})$

15. The angle through which the coordinates axes be rotated so that xy term in the equation $5x^2 + 4\sqrt{3}\,xy + 9y^2 = 0$ may be missing, is

(a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

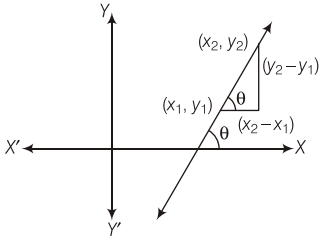
EQUATIONS OF STRAIGHT LINE IN DIFFERENT FORMS

Straight Line

A straight line can be defined as the locus of a point satisfying the condition $ax + by + c = 0$, where a, b, c are constants.

Slope (Gradient) of a Line

In Mathematics, the slope or gradient of a line is a number which is used to indicate the steepness of a line as well as indicating whether the line is tilled uphill or downhill. Slope of a line is denoted by 'm'.



If a line passing through (x_1, y_1) and (x_2, y_2) and makes an angle say 'θ' with positive direction of X-axis in anti-clockwise direction, then slope of a line is

$$m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \tan\theta$$

REMEMBER

- Slope of the line parallel to X-axis is $m = \tan 0° = 0$ and parallel to Y-axis is $m = \tan 90° = \infty$ (not defined).
- Slope of a line equally inclined with axes is 1 or -1 as it makes an angle of 45° or 135° respectively, with X-axis.
- Slope of the line $ax + by + c = 0$ is $-a/b$.
- If m_1 and m_2 are the slopes of two lines , then
  (i) lines are parallel iff $m_1 = m_2$.
  (ii) lines are perpendicular iff $m_1 m_2 = -1$.

Various Form of Equation of Straight Line

A line is determined uniquely by anyone of the following ways

(a) If two different points are known, then we can draw a unique straight line.

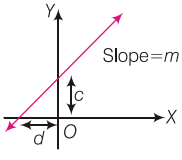
(b) If a point and its inclination or slope is given.

Let us look at the different forms of equation of straight line.

(i) **Slope form** Equation of line through the origin and having slope m is $\boldsymbol{y = mx}$.

(ii) **Point and slope form** Equation of a line through the point (x_1, y_1) and having slope m is $\boldsymbol{y - y_1 = m(x - x_1)}$.

(iii) **Slope intercept form** Equation of line with slope m and makes an intercept c on Y-axis is $\boldsymbol{y = mx + c}$.

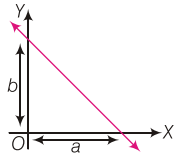


If a line makes intercept d with X-axis, then $\boldsymbol{y = m(x - d)}$.

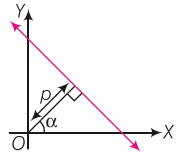
(iv) **Two points form** If (x_1, y_1) and (x_2, y_2) are the coordinates of two points, then equation of straight line is

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

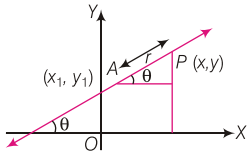
(v) **Intercept form** If a line makes intercept a and b with the coordinate axes, then equation of straight line is $\frac{x}{a} + \frac{y}{b} = 1$.



(vi) **Normal form** If p is length of perpendicular drawn from origin to straight line and this perpendicular makes angle α with positive side of X-axis, then equation of line is $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$.



(vii) **Parametric form** Equation of straight line passing through (x_1, y_1) and making an angle θ with positive direction of X-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = \pm r$, where r is distance of the point (x, y) from (x_1, y_1). This form is also known as symmetrical or distance form.



Note The coordinates of any point on this line may be taken as $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$.

Point and its Inclination with the Line is Given

The equation of straight line which passes through a given point (x_1, y_1) and make a given angle α with the given straight line, $y = mx + c$, is $y - y_1 = \left(\frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}\right)(x - x_1)$.

SPEED BOOSTER

- Equation of line, which is parallel to the line $ax + by + c = 0$, is $ax + by + k = 0$. We can find value of k by additional information given in question.
- Equation of line which is perpendicular to line $ax + by + c = 0$ is $bx - ay + k = 0$.

General Equation of Straight Line and its Transformation

General equation of straight line is $ax + by + c = 0$. General equation of straight line can be written in different forms as follows

(i) **Slope intercept form** $y = -(a/b)x - c/b$, slope $(m) = -a/b$ and intercept of Y-axis is $-c/b$.

(ii) **Intercept form** $\frac{x}{-c/a} + \frac{y}{-c/b} = 1$,

x-intercept is $-c/a$ and y-intercept is $-c/b$.

(iii) **Normal form** To change the general form of a line into normal form, first take c to right side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$ like $\frac{-ax}{\sqrt{a^2 + b^2}} + \frac{-by}{\sqrt{a^2 + b^2}} = \frac{c}{\sqrt{a^2 + b^2}}$

Then, $\cos\alpha = \frac{-a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \frac{-b}{\sqrt{a^2 + b^2}}$ and $p = \frac{c}{\sqrt{a^2 + b^2}}$

Best Practice SHOTS

16. Let $P(-1, 0), Q(0, 0)$ and $R(3, 3\sqrt{3})$ be three points. The equation of the bisector of the angle PQR, is [AIEEE 2007]

(a) $\frac{\sqrt{3}}{2}x + y = 0$ (b) $x + \sqrt{3}y = 0$

(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$

17. Let PS be the median of the triangle with vertices $P(2, 2), Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS, is [IIT JEE 2000]

(a) $2x - 9y - 7 = 0$ (b) $2x - 9y - 11 = 0$

(c) $2x + 9y - 11 = 0$ (d) $2x + 9y + 7 = 0$

18. The point $P(a, b)$ lies on the straight line $3x + 2y = 13$ and the point $Q(b, a)$ lies on the straight line $4x - y = 5$, then the equation of the line PQ is [Delhi CEE 2008]

(a) $x - y = 5$ (b) $x + y = 5$ (c) $x + y = -5$ (d) $x - y = -5$

19. If a straight line $ax + by + c = 0$ makes a triangle of constant area with coordinate axes, then

(a) a, b, c are in GP (b) a, c, b are in GP

(c) c, a, b are in GP (d) None of these

20. A line forms a triangle of area $54\sqrt{3}$ sq units with the coordinate axes. If the perpendicular drawn from the origin to the line makes an angle of $60°$ with the X-axis, then the equation of the line is

(a) $\sqrt{3}x + y = 18$ (b) $x + \sqrt{3}y = 9$

(c) $x + \sqrt{3}y = 18$ (d) None of these

21. A line through the point $P(2, 4)$ intersects the line $x + y = 9$ at the point Q. The minimum distance of PQ is

(a) $\frac{5}{\sqrt{2}}$ (b) $\frac{7}{\sqrt{2}}$ (c) $\frac{3}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

22. The point $A(2, 1)$ is translated parallel to the line $x - y = 3$ by a distance 4 units. If the new position A' is in third quadrant, then the coordinates of A' are

(a) $(2 + 2\sqrt{2}, 1 + 2\sqrt{2})$ (b) $(-2 + \sqrt{2}, -1 - 2\sqrt{2})$

(c) $(2 - 2\sqrt{2}, 1 - 2\sqrt{2})$ (d) None of these

23. The lines $y = m_1x, y = m_2x$ and $y = m_3x$ makes equal intercepts on the line $x + y = 1$, then

(a) $2(1 + m_1)(1 + m_3) = (1 + m_2)(2 + m_1 + m_3)$

(b) $(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$

(c) $(1 + m_1)(1 + m_2) = (1 + m_3)(2 + m_1 + m_3)$

(d) $2(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$

24. The equation of a line passing through $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$, is

(a) $x\cos\theta - y\sin\theta = a\cos\theta$

(b) $x\cos\theta - y\sin\theta = a\cos 2\theta$

(c) $x\cos\theta - y\sin\theta = -a\cos 2\theta$

(d) None of the above

FAMILY OF LINES AND LENGTH OF PERPENDICULAR

Equation of Family of Lines

Equation of family of lines, passing through the intersection of $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $L_1 + \lambda L_2 = 0$, where λ is any parameter.

Note Point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - b_1a_2}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) = \left(\frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right)$$

Angle between Two Lines

If $y = m_1x + c_1$ and $y = m_2x + c_2$ are two lines and θ is angle between them, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

SPEED BOOSTER

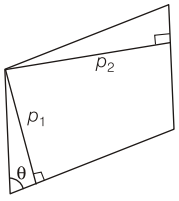
- Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two given lines.
- If θ is the angle between two given lines, then $\tan\theta = \left|\frac{b_1a_2 - a_1b_2}{a_1a_2 + b_1b_2}\right|$
- Two lines are coincident iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- Two lines are parallel iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
- Two lines intersect each other at one point iff $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
- Two lines are perpendicular iff $a_1a_2 + b_1b_2 = 0$

Length of Perpendicular from a Point on a Line

The length of perpendicular from point $P(x_1, y_1)$ to the line $ax + by + c = 0$ is $d = \left|\frac{ax + by + c}{\sqrt{a^2 + b^2}}\right|_{(x_1, y_1)} = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

SPEED BOOSTER

- If two lines are parallel such as $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$, then distance between the lines,
$$d = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$$
- The area of parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 and p_2 are distances between two pairs of opposite sides and θ is the angle between any two adjacent sides.
- If $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$, then area of parallelogram formed by these lines
$$= \left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$$



- Length of perpendicular from the origin to the line $ax + by + c = 0$ is
$$\left|\frac{a\cdot 0 + b\cdot 0 + c}{\sqrt{a^2 + b^2}}\right| = \frac{|c|}{\sqrt{a^2 + b^2}}.$$
- The length of perpendicular from (x_1, y_1) to the line $x\cos\alpha + y\sin\alpha = p$ is $|x_1\cos\alpha + y_1\sin\alpha - p|$.

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25. If a, b and c are in AP, then $ax + by + c = 0$ represents [Delhi CEE 1996]

(a) a straight line (b) a family of concurrent lines
(c) a family of parallel lines (d) None of these

26. If p and p' are the perpendiculars from the origin to the straight lines $x\sec\theta + y\,\text{cosec}\,\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, respectively, then

(a) $p^2 + p'^2 = a^2$ (b) $p^2 + 4p'^2 = a^2$
(c) $4p^2 + p'^2 = a^2$ (d) $4p^2 + p'^2 = 4a^2$

27. Angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} - \frac{y}{b} = 1$ is [MPPET 1995]

(a) $2\tan^{-1}\left(\frac{b}{a}\right)$ (b) $\tan^{-1}\frac{2ab}{a^2 + b^2}$
(c) $\tan^{-1}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)$ (d) None of these

28. For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$, then [JEE Advanced 2014]

(a) $a + b - c > 0$
(b) $a - b + c < 0$
(c) $a - b + c > 0$
(d) $a + b - c < 0$

29. The angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, is [Kerala PET 2012]

(a) $\tan^{-1}\left|\frac{a_1b_2 + a_2b_1}{a_1a_2 - b_1b_2}\right|$ (b) $\cot^{-1}\left|\frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1}\right|$
(c) $\cot^{-1}\left|\frac{a_1b_1 - a_2b_2}{a_1a_2 + b_1b_2}\right|$ (d) $\tan^{-1}\left|\frac{a_1b_1 - a_2b_2}{a_1a_2 + b_1b_2}\right|$

POSITION OF POINTS RELATIVE TO A LINE AND EQUATION OF ANGLE BISECTORS

Position of Points

If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lies on the origin side of $ax + by + c = 0$, but if the sign of $ax_1 + by_1 + c$ is opposite to that of c, then the point (x_1, y_1) lies on the non-origin side of $ax + by + c = 0$.

Note Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side of the line $ax + by + c = 0$, if $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$ and if they lie on opposite side of the line $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$.

Equation of Bisectors

Equation of the bisector of angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Acute and Obtuse Angle Bisector

(i) If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $c_1, c_2 > 0$ and $a_1b_2 \neq a_2b_1$, then there are following two cases

Case I If $a_1a_2 + b_1b_2 > 0$, then negative sign gives acute angle bisector and positive sign gives obtuse angle bisector.

Case II If $a_1a_2 + b_1b_2 < 0$, then negative sign gives obtuse angle bisector and positive sign gives acute angle bisector.

(ii) Algorithm to determine the bisector of the angle containing the origin and that of the angle not containing the origin.

***Step* I** Rewrite equations such that constant term are positive.

***Step* II** Positive (+) sign gives angle bisector, which contains the origin and negative (–) sign gives the equation of angle bisector, which does not contain the origin.

i.e. $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

where, $c_1, c_2 > 0$ gives equation which contains origin and equation

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}},$$

does not contain origin.

(iii) Algorithm to determine the bisector of the angle containing the given point

If (h, k) is any point, then the equation of angle bisector which contains a given point, we follow following steps

***Step* I** Find the value of the equations at given point.

***Step* II** If both are of the same sign, then the equation is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

and it contains the point (h, k). If value of the equations evaluated in Step I is of opposite sign, then the equation is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

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30. The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has equation $x - 7y + 5 = 0$. The equation of the other line is

(a) $3x + 3y - 1 = 0$ (b) $x - 3y + 2 = 0$
(c) $5x + 5y - 3 = 0$ (d) None of these

31. The equation of the bisector of the angle between the lines $2x + y - 6 = 0$ and $2x - 4y + 7 = 0$ which contains the point (1, 2) is

(a) $6x - 2y - 5 = 0$ (b) $2x + 6y - 19 = 0$
(c) $6x + 2y - 5 = 0$ (d) $2x + 6y + 19 = 0$

32. If (a, a^2) falls inside the angle made by the lines $y = \frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belongs to the interval [AIEEE 2006]

(a) $\left(0, \frac{1}{2}\right)$ (b) $(3, \infty)$ (c) $\left(\frac{1}{2}, 3\right)$ (d) $\left(-3, -\frac{1}{2}\right)$

33. Points (1, 2) and (–2, 1) are

(a) on the same side of the line $4x + 2y = 1$
(b) on the line $4x + 2y = 1$
(c) on the opposite side of $4x + 2y = 1$
(d) None of the above

PAIR OF STRAIGHT LINES

Homogeneous Equation

An equation of the form $a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + ... + a_nx^n = 0$ in which the sum of the powers of x and y in every term is same (i.e., n), is called a homogeneous equation (of degree n).

Now, let us look at the following facts

1. A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through origin.
   (a) If $h^2 > ab$, then lines are distinct and real.
   (b) If $h^2 = ab$, then lines are coincident.
   (c) If $h^2 < ab$, then lines are imaginary.
2. If $y = m_1x$ and $y = m_2x$ are two equations represented by $ax^2 + 2hxy + by^2 = 0$,
   then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.
3. If θ is the acute angle between the pair of straight lines $ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

Note (i) Lines are right angle to each other, if $a + b = 0$, i.e. coefficient of x^2 + coefficient of $y^2 = 0$.
(ii) Equally inclined to the X-axis is $h = 0$, i.e. coefficient of $xy = 0$.
(iii) A homogeneous equation of degree n represents n straight line passing through the origin.

General Equation of Second Degree Representing a Pair of Straight Line

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$,

i.e. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(a) Angle between the two lines representing by a general equation is $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(b) The joint equation of pair of straight lines joining origin to the points of intersection of the line given by $lx + my + n = 0$ and second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

Above equation is obtained by homogeneous equation of second degree curve with the help of $\left(\frac{lx + my}{-n}\right) = 1$.

(c) The equation to the straight lines bisecting the angle between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

(d) The product of perpendiculars drawn from (x_1, y_1) to the pair of straight lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}}$.

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- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent two parallel lines, if $g^2 - ac > 0$ and $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ and the distance between them is $2\sqrt{\frac{g^2 - ac}{a(a + b)}}$ or $2\sqrt{\frac{f^2 - ac}{b(a + b)}}$
- If the equation of a pair of straight line is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the point of intersection is given by $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

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34. Area of triangle formed by the lines $x + y = 3$ and angle bisectors of pair of straight lines $x^2 - y^2 + 2y = 1$ is [IIT JEE 2004]

(a) 2 sq units (b) 4 sq units
(c) 6 sq units (d) 8 sq units

35. The equation of pair of straight lines perpendicular to the pair $ax^2 + 2hxy + by^2 = 0$ is [MPPET 2013]

(a) $ax^2 - 2hxy + by^2 = 0$ (b) $bx^2 + 2hxy + ay^2 = 0$
(c) $ay^2 - 2hxy + bx^2 = 0$ (d) $ay^2 - bx^2 = 0$

36. Let a and b be non-zero and real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

(a) four straight lines, when $c = 0$ and a, b are of the same sign.
(b) two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a
(c) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a
(d) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

37. If the pairs of lines $x^2 + 2xy + ay^2 = 0$ and $ax^2 + 2xy + y^2 = 0$ have exactly one line in common, then the joint equation of the other two lines is given by

(a) $3x^2 + 8xy - 3y^2 = 0$
(b) $3x^2 + 10xy + 3y^2 = 0$
(c) $y^2 + 2xy - 3x^2 = 0$
(d) $x^2 + 2xy - 3y^2 = 0$

38. The condition that one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ may coincide with one of those given by the equation $a'x^2 + 2h'xy + b'y^2 = 0$ is

(a) $(ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$
(b) $(ab' - a'b)^2 = (ha' - h'a)(bh' - b'h)$
(c) $(ha' - h'a)^2 = 4(ab' - a'b)(bh' - b'h)$
(d) $(bh' - b'h)^2 = 4(ab' - a'b)(ha' - h'a)$

39. The equation $a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ represent

(a) two pairs of perpendicular straight lines
(b) two pairs of parallel straight lines
(c) two pairs of straight lines which are equally inclined to each other
(d) None of the above

40. The equation $xy + a^2 = a(x + y)$ represents [MPPET 1991]

(a) a parabola
(b) a pair of straight lines
(c) an ellipse
(d) two parallel straight lines

41. The difference of the tangents of the angles which the lines $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make with X-axis is

(a) 4 (b) 6
(c) -4 (d) 2

42. Equation of pair of straight lines drawn through (1, 1) and perpendicular to the pair of lines $3x^2 - 7xy + 2y^2 = 0$ is [Delhi CEE 2002]

(a) $2x^2 + 7xy - 11x + 6 = 0$
(b) $2(x - 1)^2 + 7(x - 1)(y - 1) - 3y^2 = 0$
(c) $2(x - 1)^2 + 7(x - 1)(y - 1) - 3(y - 1)^2 = 0$
(d) None of the above

43. If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}\left(\frac{1}{3}\right)$, where '$\lambda$' is a non-negative real number. Then, λ is equal to [Odisha JEE 2002]

(a) 2 (b) 0
(c) 3 (d) 1

44. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the product of the perpendicular from the origin on these straight lines, is

(a) $\frac{c}{\sqrt{(a-b)^2 - 4h^2}}$ (b) $\frac{c}{\sqrt{(a-b)^2 + 4h^2}}$
(c) $\frac{c}{\sqrt{(a+b)^2 - 4h^2}}$ (d) None of these

45. If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1} 3$, where λ is a non-negative real number. Then, λ is equal to [Odisha JEE 2002]

(a) 2 (b) 0
(c) 3 (d) 1

46. The equation of pair of lines joining origin to the points of intersection of $x^2 + y^2 = 9$ and $x + y = 3$ is [WB JEE 2006]

(a) $x^2 + (3 - x)^2 = 9$ (b) $xy = 0$
(c) $(3 + y)^2 + y^2 = 9$ (d) $(x - y)^2 = 9$

47. The lines $(lx + my)^2 - 3(mx - ly)^2 = 0$ and $lx + my + n = 0$ form

(a) an isosceles triangle
(b) a right angled triangle
(c) an equilateral triangle
(d) None of the above

48. The equation $2x^2 - 3xy - py^2 + x + qy - 1 = 0$ represents two mutually perpendicular lines, if

(a) $p = 3, q = 2$ (b) $p = -2, q = -3$
(c) $p = -2, q = 3$ (d) $p = 2, q = -\frac{9}{2}$

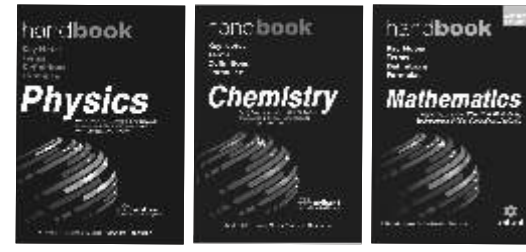
49. The product of the perpendiculars drawn from the point (1, 2) to the pair of lines $x^2 + 4xy + y^2 = 0$ is

(a) 9/4 (b) 3/4
(c) 13/4 (d) 9/16

50. If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of the another sector, then [AIEEE 2005]

(a) $3a^2 + 2ab + 3b^2 = 0$ (b) $3a^2 + 10ab + 3b^2 = 0$
(c) $3a^2 - 2ab + 3b^2 = 0$ (d) $3a^2 - 10ab + 3b^2 = 0$

handbook
Key Notes
Terms
Definitions
Formulae



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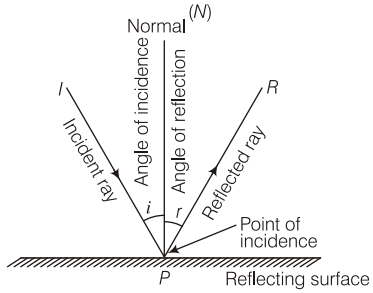
REFLECTION

Reflection of Light

When a ray of light falls on a smooth polished surface (Mirror) separating two media, a part of it is reflected back into the first medium.



IP is the incident ray and PR is the reflected ray. A perpendicular drawn to the surface, at the point of incidence P is called the normal. Hence, PN is the normal. The angle between the incident ray and the normal ($\angle IPN$) is called the angle of incidence which is represented by $\angle i$.

i.e. $\angle IPN = \angle i$ = Angle of incidence and the angle between the reflected ray and the normal ($\angle RPN$) is called the angle of reflection which is represented by $\angle r$.

i.e. $\angle RPN = \angle r$ = Angle of reflection.

Laws of Reflection

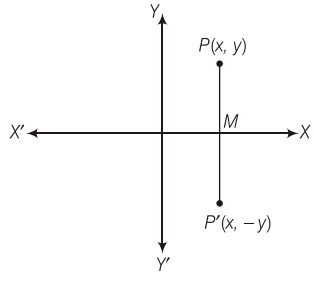
(i) The incident ray, normal and the reflected ray to a surface at the point of incidence all lie in the same plane.

(ii) The angle of incidence = angle of reflection, i.e. $\angle i = \angle r$

Reflection of a Point under Different Conditions

(I) Reflection of a Point with Respect to *X*-axis

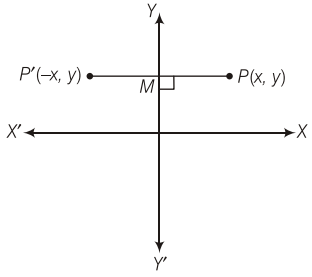
If $P(x, y)$ is any point, then its reflection about X-axis (i.e. in line $y = 0$) will be $P'(x, -y)$.



Note The reflection of the line $ax + by + c = 0$ about X-axis is $ax - by + c = 0$

(II) Reflection of a Point with Respect to *Y*-axis

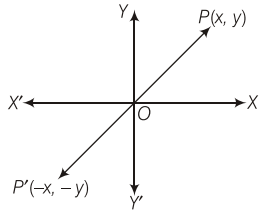
If $P(x,y)$ is any point, then its reflection about Y-axis (i.e. in line $x = 0$) will be $P'(-x, y)$.



Note The reflection of the line $ax + by + c = 0$ about Y-axis is $-ax + by + c = 0$.

(III) Reflection of a Point with Respect to Origin

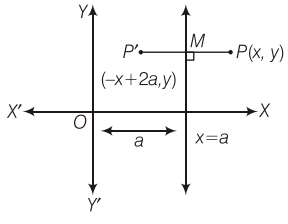
If $P(x, y)$ is any point, then its reflection about the origin will be $P'(-x, -y)$.



Note The reflection of the line $ax + by + c = 0$ about the origin is $-ax - by + c = 0$.

(IV) Reflection of a Point with Respect to the Line $x = a$

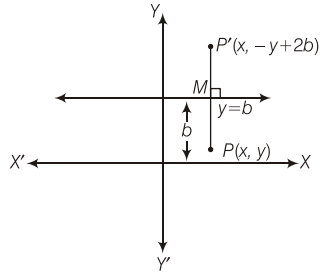
If $P(x, y)$ is any point, then its reflection about the line $x = a$, will be $P'(-x + 2a, y)$.



Note The reflection of the line $ax + by + c = 0$ about the line $x = \lambda$ is $a(2\lambda - x) + by + c = 0$.

(V) Reflection of a Point with Respect to the Line $y = b$

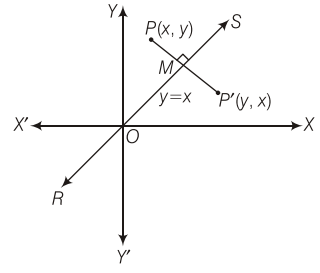
If $P(x, y)$ is any point, then its reflection about the line $y = b$, will be $P'(x, -y + 2b)$.



Note The reflection of the line $ax + by + c = 0$ about the line $y = \mu$ is $ax + b(2\mu - y) + c = 0$.

(VI) Reflection of a Point with Respect to the Line $y = x$

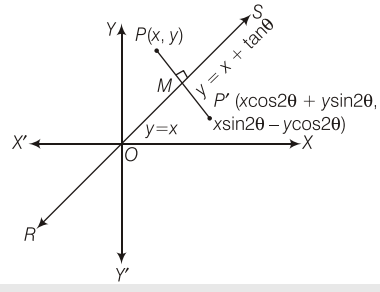
If $P(x, y)$ is any point, then its reflection about the line $y = x$, is $P'(y, x)$.



Note The reflection of the line $ax + by + c = 0$ about the line $y = x$ is $ay + bx + c = 0$.

(VII) Reflection of a Point with Respect to the Line $y = x\tan\theta$

If $P(x, y)$ is any point, then its reflection about the line $y = x\tan\theta$, is $P'(x\cos 2\theta + y\sin 2\theta, x\sin 2\theta - y\cos 2\theta)$.



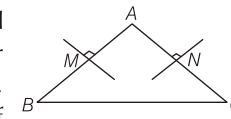
Note (i) If (x_1, y_1) is any point and $ax + by + c = 0$ is equation of line, then to find the coordinates of foot of perpendicular use the result $\frac{(x_2 - x_1)}{a} = \frac{y_2 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$. By simplifying the above result, we can find the values of x_2 and y_2.

(ii) Image of reflection of point (x_1, y_1) about a line mirror $ax + by + c = 0$ is $\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$

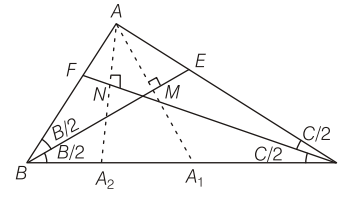
Use of Reflection

To make the following types of problems easier, we use the reflection

Type I If vertex of a ΔABC and equations of perpendicular bisectors of AB and AC are given, then B and C are the reflections of A about the perpendicular bisectors of AB and AC. (where, M and N are the mid-points of AB and AC).

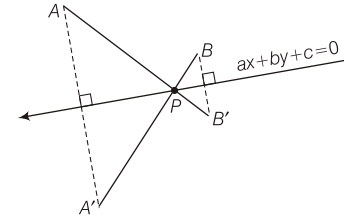


Type II The reflection of vertex A of a ΔABC about the angular bisector of angles B and C lie on the side BC. (By congruence) A_1 and A_2 are the images of A about the angle bisectors BE and CF respectively, where M and N are the mid-points of AA_1 and AA_2.



Optimisation
(Minimisation or Maximisation)

(i) **Minimisation** Let A and B are two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $PA + PB$ is minimum. Then, find the image of A or B about the line $ax + by + c = 0$ (say A' or B') then join B' with A or A' with B and wherever it intersects $ax + by + c = 0$ is the required point.

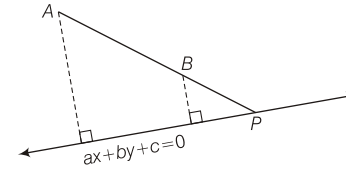


$\therefore$ $PA + PB = PA + PB'$

or $PA + PB = PA' + PB$

and $|PA + PB| = |PA + PB'| = |AB'|$ [minimum value]

(ii) **Maximisation** Let A and B be two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $|PA - PB|$ is maximum, then find the equation of line joining A and B and the point at which it intersects $ax + by + c = 0$ is the required point.



and $|PA - PB| = |AB|$ [maximum value]

Best Practice SHOTS

51. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching X-axis, then the equation of the reflected ray is

(a) $y = x + \sqrt{3}$ (b) $\sqrt{3}y = x - \sqrt{3}$
(c) $y = \sqrt{3}x - \sqrt{3}$ (d) $\sqrt{3}y = x - 1$

52. A light ray emerging from the point source placed at $P(1, 3)$ is reflected at a point Q in the X-axis. If the reflected ray passes through the point $R(6, 7)$, then the abscissa of Q is [JEE Main 2013]

(a) 1 (b) 3 (c) 7/2 (d) 5/2

53. If A and B are $(4, 2)$ and $(2, 4)$ respectively, then a point P on the line $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum, is

(a) (22, 28) (b) (28, – 22) (c) (–22, 28) (d) (28, 22)

54. The image of the point $(-2, -7)$ under the transformation $(x, y) \to (x - 2y, -3x + y)$ is

(a) (12, – 1) (b) (1, 12) (c) (–1, 12) (d) None

55. A light beam, emanating from the point $(3, 10)$ reflects from the straight line $2x + y - 6 = 0$ and then passes through the point $(7, 2)$. The equations of the incident and reflected beams, are respectively

(a) $3x - y + 1 = 0,\ x + 13y - 3 = 0$
(b) $3x - y + 1 = 0,\ x + 3y - 13 = 0$
(c) $x - 3y + 1 = 0,\ x + 3y - 13 = 0$
(d) None of the above

Answers

1. (b)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (d)
11. (a)	**12.** (b)	**13.** (a)	**14.** (d)	**15.** (a)	**16.** (c)	**17.** (d)	**18.** (b)	**19.** (b)	**20.** (c)
21. (c)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (a)	**29.** (b)	**30.** (c)
31. (a)	**32.** (c)	**33.** (c)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (a)	**39.** (c)	**40.** (b)
41. (d)	**42.** (d)	**43.** (a)	**44.** (b)	**45.** (b)	**46.** (b)	**47.** (c)	**48.** (d)	**49.** (c)	**50.** (a)
51. (b)	**52.** (d)	**53.** (c)	**54.** (a)	**55.** (b)					

Formulae at a Glance

1. General equation of straight line is $ax + by + c = 0$, whose slope is $m = -a / b$.

2. Equations of straight lines in different forms are following

(i) Slope intercept form is $y = mx + c$
(ii) Point slope form is $y - y_1 = m(x - x_1)$
(iii) Intercept form is $\frac{x}{a} + \frac{y}{b} = 1$
(iv) Two points form is $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
(v) Normal form is $x\cos\alpha + y\sin\alpha = p$
(vi) Distance form is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$

3. (i) Equation of a line which is parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$.
(ii) Equation of line which is perpendicular to $ax + by + c = 0$ is $bx - ay + \lambda = 0$.

4. If the equation of line is $a\sin\theta + b\cos\theta = c$, then line

(i) parallel to it, is $a\sin\theta + b\cos\theta = d$
(ii) perpendicular to it, is $a\sin\left(\frac{\pi}{2} + \theta\right) + b\cos\left(\frac{\pi}{2} + \theta\right) = d$

5. Point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

6. (i) Angle between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

If $m_1m_2 = -1$, then lines are perpendicular and if $m_1 = m_2$, then lines are parallel.

(ii) Angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by $\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$.

7. Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (ii) parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
(iii) perpendicular, if $a_1a_2 + b_1b_2 = 0$ (iv) intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

8. Equation of line passing through the point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$.

9. Equation of bisectors of the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

(i) If $a_1a_2 + b_1b_2 > 0$, then '+ ve' sign gives obtuse angle bisector and '– ve' sign gives acute angle bisector.
(ii) If $a_1a_2 + b_1b_2 < 0$, then '+ ve' sign gives acute angle bisector and '– ve' sign gives obtuse angle bisector.

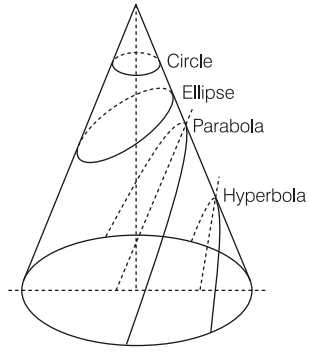
10. (i) Length of perpendicular from the point (x_1, y_1) to the line $ax + by + c = 0$ is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

(ii) Length of perpendicular from the point (x_1, y_1) to the line $x\cos\alpha + y\sin\alpha\ p$ is $|x_1\cos\alpha + y_1\sin\alpha - p|$.
(iii) Distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

To be continued at page 36

Conic Sections

Conic section is important topic for JEE Main and Advanced. Every year atleast 3 questions are asked in JEE Main and Advanced. Most frequent topics that asked, are basic of conics, tangent and normal to conics, chord of conics, position of point and chord of contact. Students are advised to memorise maximum result to enhance speed in the exams. There are many similarity in problem solving and in conceptual approach of various types of conics. If you prepare one section of cone such as parabola, then it will help you to understand other section of cone.



A conic section or conic is the locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) always bears a constant ratio to its distance from a fixed line (i.e. directrix). This constant ratio is called eccentricity of the conic and it is denoted by '*e*'.

General Equation of Conic

A second degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents degenerate and non-degenerate conic section.

1. Degenerate Conic Section

When the focus lies on the directrix, then above equation represents pair of straight lines, if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Now, we have the following cases

(i) If $e > 1$ or $h^2 > ab$, then the lines will be real and distinct intersecting at fixed point.

(ii) If $e = 1$ or $h^2 = ab$, then the lines will be coincident (parallel).

(iii) If $e < 1$ or $h^2 < ab$, then the lines will be imaginary.

2. Non-degenerate Conic Section

When the focus does not lie on the directrix, then above equation represent

(i) Circle : $a = b$, $h = 0$ and $\Delta \neq 0$ or $e = 0$.

(ii) Parabola : $h^2 = ab$ and $\Delta \neq 0$ or $e = 1$.

(iii) Ellipse : $h^2 < ab$ and $\Delta \neq 0$ or $0 < e < 1$.

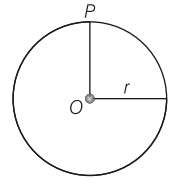
(iv) Hyperbola : $h^2 > ab$ and $\Delta \neq 0$ or $e > 1$.

(v) Rectangular hyperbola : $a + b = 0$, $\Delta \neq 0$ and $h^2 > ab$.

CIRCLE

Circle as a Locus

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point always remains constant. This fixed point is called centre of circle and the constant distance is called radius of circle.



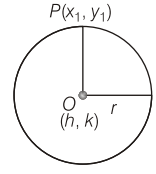
Equation of Circle in Various Form

1. Centre-Radius Form

If $P(x_1, y_1)$ is any point on the circle and r is the radius of the circle, then

(a) Equation of a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

(b) Equation of a circle with centre $(0, 0)$, i.e. at origin and radius r is $x^2 + y^2 = r^2$.



2. Parametric Form

The parametric equation of the circle $x^2 + y^2 = a^2$ is

$x = a\cos\theta$ and $y = a\sin\theta$, where θ is a parameter.

Note (i) The parametric equations of a circle

$(x - h)^2 + (y - k)^2 = a^2$ is

$x = h + a\cos\theta$ and $y = k + a\sin\theta$

(ii) The equation of the chord of a circle $x^2 + y^2 = a^2$ joining $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is

$$x\cos\left(\frac{\alpha + \beta}{2}\right) + y\sin\left(\frac{\alpha + \beta}{2}\right) = a\cos\left(\frac{\alpha - \beta}{2}\right).$$

3. General Form

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where g, f and c are constants, $(-g, -f)$ be centre and $r = \sqrt{g^2 + f^2 - c}$ be the radius of a circle.

REMEMBER

- Nature of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ has following possible cases
  (a) If $g^2 + f^2 - c > 0$, then real circle is possible.
  (b) If $g^2 + f^2 - c = 0$, then radius of the circle is zero. Hence, circle is called point circle.
  (c) If $g^2 + f^2 - c < 0$, then radius of the circle will be imaginary number. Hence, in this case circle is called imaginary circle.
- Two circles having same centre but different radii are called concentric circles. Hence, equation of concentric circle differ by only radius. If (h, k) is centre and r_1, r_2 are radii of the concentric circles, then $(x-h)^2 + (y-k)^2 = r_1^2$ and $(x-h)^2 + (y-k)^2 = r_2^2$.
- To find centre and radius of the circle, use following steps

***Step* I** Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.

***Step* II** Coordinates of centre $(-g, -f)$ given by

$g = \frac{1}{2}$(Coefficient of x) and $f = \frac{1}{2}$(Coefficient of y)

and radius is $\sqrt{g^2 + f^2 - \text{Constant term}}$.

4. Diametric Form

The equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as its diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

5. Equation of Circle Passing through Three Non-collinear Points

If three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then their coordinates satisfy its equation, i.e.

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0,$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$
and
$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

On solving these equations, we can obtain the value of g, f and c and put these values in the equation of circle to get the required equation of circle.

Intercept Made by a Circle on the Axes

The intercepts made by a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

(a) on X-axis is $2\sqrt{g^2 - c}$, where $g^2 > c$.

(b) on Y-axis is $2\sqrt{f^2 - c}$, where $f^2 > c$.

MATHS MAXIMA

- If $g^2 > c \Rightarrow$ Circle cuts the X-axis at two distinct points.
- If $g^2 = c \Rightarrow$ Circle touches the X-axis.
- If $g^2 < c \Rightarrow$ Circle lies completely above or below the X-axis.

Similar in the case for intercepts on Y-axis.

Position of a Point with Respect to a Circle

Let equation of a circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

If value of a circle at $P(x_1, y_1)$ is $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, then following cases arise, which are as

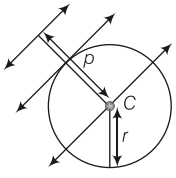
(a) If $S_1 = 0$, then point lies on the circle.
(b) If $S_1 > 0$, then point lies outside the circle.
(c) If $S_1 < 0$, then point lies inside the circle.

Note Maximum and minimum distance of the point P from the circle is $|r + PC|$ and $|r - PC|$, where C and r is the centre and radius of a circle, respectively.

Line and Circle

Let $L = 0$ and $S = 0$ be the equation of line and circle, respectively. If r is the radius and p is the length of the perpendicular from the centre of circle to the line $L = 0$, then the following cases are arise

(a) If $p > r$, then the line does not meet the circle, i.e. it passes outside.
(b) If $p = r$, then the line touches the circle.
(c) If $p < r$, then the line is secant to the circle.
(d) If $p = 0$, then the line is a diameter of the circle.



Tangent to a Circle

If a line touches the circle at a point, then the line is called tangent to the circle and the point, where it touches the circle, is called the point of contact.

Various Form of the Equation of Tangent

1. Cartesian Form

If (x_1, y_1) is the point of contact, then the equation of tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note If (x_1, y_1) is the point of contact, then the equation of tangent to the circle $(x-h)^2 + (y-k)^2 = r^2$ is $(x-h)(x-x_1) + (y-k)(y-y_1) = r^2$.

2. Parametric Form

If $(r\cos\theta, r\sin\theta)$ are the parametric coordinates of the circle $x^2 + y^2 = r^2$, then the equation of tangent is $x\cos\theta + y\sin\theta = r$.

Note The point of intersection of tangents drawn to the circle at the points $P(\theta)$ and $Q(\alpha)$ is $\left[\frac{r\cos\left(\frac{\theta+\alpha}{2}\right)}{\cos\left(\frac{\theta-\alpha}{2}\right)}, \frac{r\sin\left(\frac{\theta+\alpha}{2}\right)}{\cos\left(\frac{\theta-\alpha}{2}\right)}\right]$

3. Slope Form

The equation of a tangents of slope m to the circle $x^2 + y^2 = r^2$ is $y = mx \pm r\sqrt{1+m^2}$.

Note The condition that a line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ is $c = \pm r\sqrt{1+m^2}$ and point of contact is $\left(\pm\frac{r^2m}{c}, \mp\frac{r^2}{c}\right)$.

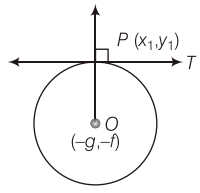
REMEMBER

- If slope of the tangent is given, then two parallel tangents can be drawn to the circle at the end of diameter.
- Equation of tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is
$$y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}.$$
- If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then the point of contact is given by $\left(-\frac{ar^2}{c}, \frac{-br^2}{c}\right)$.

Normal to a Circle

The normal of a circle at any point is a straight line which is perpendicular to the tangent line at the point of contact.

Note Normal to a circle always passes through the centre.



Various Form of the Equation of Normal to a Circle

1. Point Form

If $P(x_1, y_1)$ is the point of contact, then the equation of normal to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

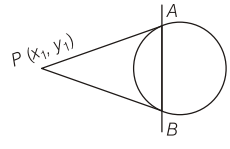
$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1) \quad \text{or} \quad \left(\frac{y - y_1}{y_1 + f}\right) = \left(\frac{x - x_1}{x_1 + g}\right)$$

2. Parametric Form

If $(r\cos\theta, r\sin\theta)$ is the parametric coordinates of a circle $x^2 + y^2 = r^2$, then the equation of normal at $(r\cos\theta, r\sin\theta)$ is $y = x\tan\theta$.

Equation of Pair of Tangent

The equation of pair of tangents drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $T^2 = SS_1$.



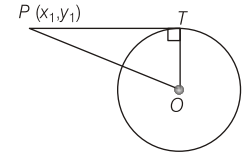
where, $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Length of the Tangent

If $P(x_1, y_1)$ is any point lies outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then length of the tangent is

$$PT = \sqrt{S_1}$$

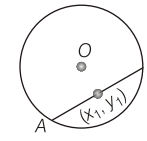


where, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note To find the length of the tangent from a point, make the coefficient of x^2 and y^2 equal to 1.

Equation of Chord Bisected at a Given Point

If $P(x_1, y_1)$ bisects the chord AB of a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of chord AB is $T = S_1$

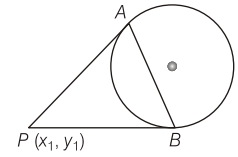


where, $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

and $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Equation of Chord of Contact

If two tangents AP and BP are drawn from an external point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact AB is $T = 0$



where, $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

REMEMBER

- The length of chord of contact is $2\sqrt{r^2 - p^2}$, where p is length of perpendicular from centre to the chord and r is radius of the circle.
- If $S = x^2 + y^2 - r^2$, then area of ΔABP is given by $\frac{r(x_1^2 + y_1^2 - r^2)^{3/2}}{x_1^2 + y_1^2}$ or $\frac{r S_1\sqrt{S_1}}{(S_1 + r^2)}$.

Diameter

The locus of the middle points of a system of parallel chords of a conic is called a diameter of the circle.

The equation of the diameter bisecting parallel chord $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = r^2$ is $x + my = 0$.

Common Tangents to Two Circles

Two circles may intersect, touch or neither intersect nor touch according to the following condition, which are given below

Let the centres and radii of two circles be C_1, C_2 and r_1, r_2 respectively.

(i) Circles neither intersect nor touch each other, when $C_1C_2 > r_1 + r_2$. In this case there are four common tangents. Two of them are Transverse Common Tangent (TCT) and other two are Direct Common Tangent (DCT).

(ii) Two circles intersect each other when $|r_1 - r_2| < C_1C_2 < |r_1 + r_2|$. In this case there are two common tangents and both of them are DCT.

(iii) Two circles touch each other

(a) externally when $C_1C_2 = r_1 + r_2$. In this case, there are three common tangents possible. Two of them are DCT and one is TCT.

(b) internally when $C_1C_2 = r_1 - r_2$. In this case one common tangent is possible, which is DCT.

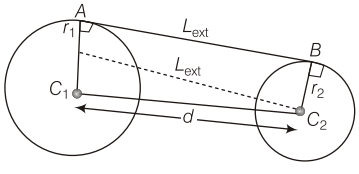
Note (i) DCT meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

(ii) TCT meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

Length of DCT and TCT

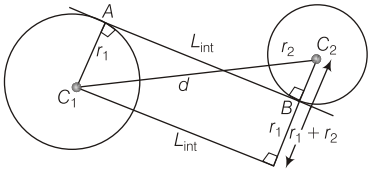
(a) If AB is external common tangent (DCT), then the length of external common tangent (or DCT) is

$$L_{ext} = L_{DCT} = \sqrt{d^2 - (r_1^2 - r_2^2)}$$



(b) If AB is internal common tangent (TCT), then the length of internal common tangent (or TCT) is

$$L_{int} = L_{TCT} = \sqrt{d^2 - (r_1 + r_2)^2}$$



SPEED BOOSTER

- The length of tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{c - c_1}$.
- If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b) meets the coordinate axes at the points A, B and O is the origin. Then, area of ΔAOB is $\frac{r^4}{2ab}$.
- If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and let P be the centre of the circle, then the area of the quadrilateral $OAPB$ is $\sqrt{c(g^2 + f^2 - c)}$.
- If the circles $x^2 + y^2 + 2gx + c^2 = 0$ and $x^2 + y^2 + 2fy + c^2 = 0$ touch each other, then $\frac{1}{g^2} + \frac{1}{f^2} = \frac{1}{c^2}$.
- The angle of intersection of two circles is defined as the angle between the tangents or angle between the normals to the two circles at their point of intersection, is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

where, d is the distance between both the centres of the circle.

Common Chord

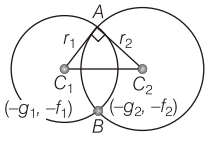
Let $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ be two given circles. Then, the chord joining the points of intersection of these circles is called their common chord and its equation is given by $S_1 - S_2 = 0$.

SPEED BOOSTER

- Length of common chord $= 2\sqrt{(C_1P)^2 - (C_1M)^2}$, where C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.
- Length of common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ is $\frac{2ab}{\sqrt{a^2 + b^2}}$.

Orthogonal Circle

Two circles $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$, are said to be orthogonal, if the tangents to the circles at each point of intersection are at 90°.



Note (i) The condition for two circles to be orthogonal is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

(ii) If circles having radii r_1 and r_2 intersect orthogonally, then length of their common chord is $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Director Circle

The locus of the point of intersection of two perpendicular tangents to a conic is called director circle.

The director circle of a circle is concentric circle, whose radius is $\sqrt{2}$ times the radius of the given circle.

Equation of Director Circle

(a) If $x^2 + y^2 = r^2$ is the equation of circle, then equation of director circle is $x^2 + y^2 = 2r^2$.

(b) If $x^2 + y^2 + 2gx + 2fy + c$ is the equation of the circle, then equation of director circle is

$$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$$

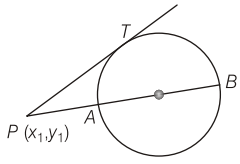
Power of Point

If $P(x_1, y_1)$ is any point outside the circle, then the power of point $P(x_1, y_1)$ with respect to circle is $PA \cdot PB$.

$\because PA \cdot PB = PT^2 = (\sqrt{S_1})^2$

Hence, power of point to a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is



$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the tangent segments drawn from it to the two circles are of equal length.

Equation of Radical Axis

The equation of radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$, i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

REMEMBER

- If two circles intersect or touch each other, then the radical axis is the common chord or common tangent of two circles, respectively.
- Radical axis is always perpendicular to but need not always bisects the line joining the centres of two circles.
- Radical axis bisects a common tangent of two circles.
- The common point of intersection of the radical axes of three circles, whose centres are non-collinear, taken two at a time is called the radical centre of three circles.

Family of Circles

Important result related to family of circles are given as

(i) If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ are two intersecting circle, then the equation of family of circles passing through the points of intersection of two given circles is $S + \lambda S_1 = 0$, where $\lambda \neq -1$.

(ii) If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle which is intersected by a line $L \equiv ax + by + c = 0$ at two real and distinct points, then the equation of family of circles passing through the points of intersection of circle and line is $S + \lambda L = 0$.

(iii) The equation of family of circles passing through (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \text{ where}$$

λ is a parameter.

(iv) The equation of family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + k[y - y_1 - m(x - x_1)] = 0$.

Two particular cases are as follow

(a) If line passes through (x_1, y_1) and parallel to Y-axis, then $(x - x_1)^2 + (y - y_1)^2 + k(x - x_1) = 0$.

(b) If line passes through (x_1, y_1) and parallel to X-axis, then $(x - x_1)^2 + (y - y_1)^2 + k(y - y_1) = 0$.

SPEED BOOSTER

- Equation of circle circumscribing a triangle whose sides are $L_1 = 0, L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + k_1L_2L_3 + k_2L_3L_1 = 0$ provided, coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.
- Equation of circle circumscribing a quadrilateral whose sides in order are represented by $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$ is $L_1L_3 + kL_2L_4 = 0$ provided, coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Coaxial System of Circles

A system of circles is said to be coaxial system of circles, if every pair of the circles in the system has the same radical axis. Now, let us consider the following cases

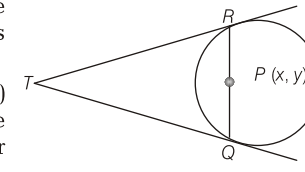
(a) If the equation of a member of a system of coaxial circles is $S = 0$ and the equation of the common radical axis is $L = 0$, then the equation representing the coaxial system of circle is

$S + \lambda L = 0$, where $\lambda \in R$.

(b) The equation of a coaxial system of circles of which two members are $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$, $(\lambda \neq -1)$ or $S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \lambda(S_1 - S_2) = 0$, where $\lambda \in R$.

Pole and Polar

Let P be a point lying within or outside a given conic. Suppose any straight line drawn through P intersects the conic at Q and R. Then, the locus of the point of intersection of the tangents to the conic at Q and R is called the polar of the given point P with respect to the given conic and the point P is called the pole of the polar.



The polar of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$ or $T = 0$.

1. Conjugate Points

Two points A and B are conjugate points with respect to a given conic, if each lies on the polar of the other.

2. Conjugate Lines

If two lines be such that the pole of one line lies on the other, then they are called conjugate lines with respect to the given conic.

Best Practice SHOTS

1. The lines $2x - 3y - 5 = 0$ and $3x - 4y = 7$ are diameters of a circle of area $154 (= 49\pi)$ sq units, then the equation of the circle is **[AIEEE 2006]**

(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 + 2x - 2y - 47 = 0$
(c) $x^2 + y^2 - 2x + 2y - 47 = 0$
(d) $x^2 + y^2 - 2x + 2y - 62 = 0$

2. If the abscissa and ordinates of two points P and Q are the roots of the equations $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$, respectively, then the equation of the circle with PQ as diameter, is

(a) $x^2 + y^2 - 2ax + 2py - b^2 - q^2 = 0$
(b) $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$
(c) $x^2 + y^2 - 2ax - 2py - b^2 - q^2 = 0$
(d) $x^2 + y^2 - 2ax - 2py + b^2 + q^2 = 0$

3. If the circle touched by $y = x$, has its centre on the positive direction of X-axis and cuts-off a chord of length 2 units along the line $\sqrt{3}y - x = 0$, then the equation of the circle is

(a) $x^2 + y^2 - 4x + 2 = 0$ (b) $x^2 + y^2 + 4x + 2 = 0$
(c) $x^2 + y^2 + 4x + 8 = 0$ (d) $x^2 + y^2 - 4x + 8 = 0$

4. The circle $x^2 + y^2 - 6x - 10y + k = 0$ is neither touch nor intersect the coordinate axes and the point (1, 4) is inside the circle. Find the range of the values of k.

(a) [25, 29] (b) (25, 29] (c) [25, 29) (d) (25, 29)

5. If $\frac{x}{\alpha} + \frac{y}{\beta} = 1$ touches the circle $x^2 + y^2 = a^2$, then the point $\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ lies on **[Odisha JEE 2005]**

(a) a straight line (b) a circle (c) a parabola (d) an ellipse

6. The locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ at points whose parametric angles differ by $\frac{\pi}{3}$, is

(a) $x^2 + y^2 = 4a^2$ (b) $3(x^2 + y^2) = a^2$
(c) $3(x^2 + y^2) = 4a^2$ (d) $4(x^2 + y^2) = 3a^2$

7. If the distances from the origin to the centres of three circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$ $(i = 1, 2, 3)$ are in GP, then the lengths of the tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in

(a) AP (b) GP (c) HP (d) None

8. If the line $2x - y + 1 = 0$ is a tangent to the circle at point (2, 5) and the centre of the circle is on the line $x + y = 9$, then the equation of the circle is

(a) $x^2 + y^2 - 12x + 6y + 5 = 0$
(b) $x^2 + y^2 - 12x - 6y - 5 = 0$
(c) $x^2 + y^2 + 12x + 6y + 15 = 0$
(d) $x^2 + y^2 - 12x - 6y + 25 = 0$

9. If the tangents are drawn from a point on the circle $x^2 + y^2 - 4x + 6y - 37 = 0$ to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$, then the angle between the tangents is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

10. For the given circles $x^2 + y^2 - 6x - 2y + 1 = 0$ and $x^2 + y^2 + 2x - 8y + 13 = 0$, which of the following is true? **[Delhi CEE 2008]**

(a) One circle lies inside the other
(b) One circle lies completely outside the other
(c) Two circles intersect in two points
(d) They touch each other externally

11. The value of k, so that $x^2 + y^2 + kx + 4y + 2 = 0$ and $2(x^2 + y^2) - 4x - 3y + k = 0$ cut orthogonally is **[Delhi CEE 2007]**

(a) $\frac{10}{3}$ (b) $-\frac{8}{3}$ (c) $\frac{-10}{3}$ (d) $\frac{8}{3}$

12. The radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2(x^2 + y^2) + 3x + 2y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, if

(a) $g = \frac{4}{3}$ and $f = 2$ (b) $g = \frac{4}{3}$ and $f = \frac{1}{2}$
(c) $g = \frac{-3}{4}$ and $f = 2$ (d) $g = \frac{3}{4}$ and $f = \frac{1}{2}$

13. The equation of the circle passing through $(1, -3)$ and the points common to the two circles $x^2 + y^2 - 6x + 8y - 16 = 0$, $x^2 + y^2 + 4x - 2y - 8 = 0$, is

(a) $x^2 + y^2 - 4x + 6y + 24 = 0$
(b) $2x^2 + 2y^2 + 3x + y - 20 = 0$
(c) $3x^2 + 3y^2 - 5x + 7y - 19 = 0$
(d) None of the above

14. Tangents drawn from the point (4, 3) to the circle $x^2 + y^2 - 2x - 4y = 0$ are inclined at an angle **[Delhi CEE 2009]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

15. The coordinates of the middle point of the chord intercepted on line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$ are

(a) $\left(\frac{-l}{l^2 + m^2}, \frac{-m}{l^2 + m^2}\right)$ (b) $\left(\frac{-ln}{l^2 + m^2}, \frac{-mn}{l^2 + m^2}\right)$
(c) $\left(\frac{-l}{n(l^2 + m^2)}, \frac{-m}{n(l^2 + m^2)}\right)$ (d) None of these

16. If the chord of contact of tangents drawn from a point (α, β) to the circle $x^2 + y^2 = a^2$ subtends a right angle at the centre of the circle, then

(a) $\alpha^2 + \beta^2 = a^2/2$ (b) $\alpha^2 + \beta^2 = a^2$
(c) $\alpha^2 + \beta^2 = 2a^2$ (d) None of these

PARABOLA

Parabola as a Locus

A parabola is the locus of a point which moves in a plane in such a way that its distance from a fixed point (i.e. focus) is equal to its distance from a fixed straight line (i.e. directrix).

Standard Equation of Parabola

Let S and QN be the focus and directrix of the parabola, respectively. If P is any point on the parabola and $PN \perp QN$, then by definition

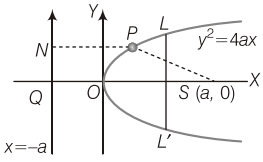
$\Rightarrow \quad PS = PN \quad \Rightarrow \quad (PS)^2 = (PN)^2$

[on squaring both sides]

$\Rightarrow \quad (x - a)^2 + y^2 = (x + a)^2$

$\Rightarrow \quad y^2 = 4ax$

which is the required equation of parabola.



Parametric Equations of Parabola

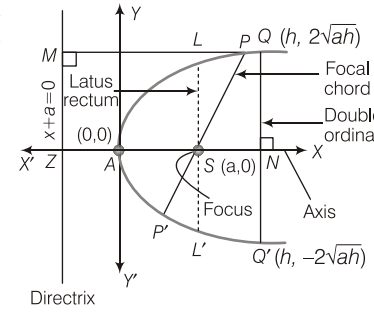
The parametric equations of parabola

$y^2 = 4ax$ are $x = at^2$ and $y = 2at$

where, t is the parameter.

Some Important Terms of Parabola

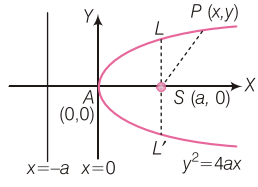
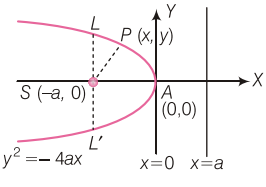
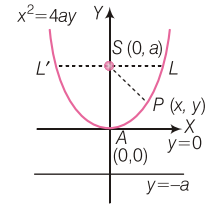
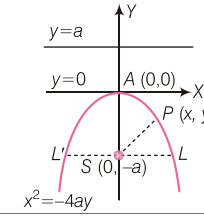
1. **Axis** The axis of the parabola is the straight line which passes through focus and perpendicular to the directrix of the parabola.
   For parabola $y^2 = 4ax$, X-axis is the axis of parabola.
2. **Symmetry** On replacing y by $-y$, then the equation $y^2 = 4ax$ remains same. So, the curve is symmetrical about X-axis .
3. **Vertex** The point of intersection of parabola and its axis is called vertex of parabola.
   For parabola $y^2 = 4ax$, $A(0, 0)$ (i.e. origin) is the vertex.
4. **Ordinate and double ordinate** Let Q be any point on the parabola.
   From Q, draw $QN \perp AX$, then QN is called ordinate and QNQ' is called double ordinate of the point Q.



5. **Focal chord** A chord of a parabola passing through its focus is called focal chord.
6. **Latusrectum** The focal chord perpendicular to the axis is called latusrectum.
   If LL' is the latusrectum of a parabola, then this line is perpendicular to AX and passing through S. Also, $L \equiv (a, 2a)$, $L' \equiv (a, -2a)$ and length of latusrectum $= LL' = 4a$
7. **Focal distance** The distance of any point $P(x, y)$ from its focus S is known as its focal distance (SP).
   According to the definition of parabola, $SP = PM = x + a$

Standard Forms of the Parabola

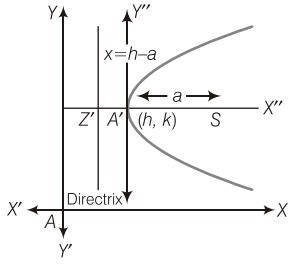
Four standard forms of the parabola are shown below along with their properties.

Standard equations	$y^2 = 4ax (a > 0)$	$y^2 = -4ax (a > 0)$	$x^2 = 4ay (a > 0)$	$x^2 = -4ay (a > 0)$
Shape of the parabola				
Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latusrectum	$4a$	$4a$	$4a$	$4a$
Equation of latusrectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity	1	1	1	1

Generalised Form of a Parabola

When origin is shifted at $A'(h, k)$ without changing the direction of the axes, then the equation of a parabola is

$$(y-k)^2 = 4a(x-h)$$



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- Axis $A'X'' \| AX$ with its vertex at $A'(h, k)$
- The focus and length of latusrectum of parabola $(y-k)^2 = 4a(x-h)$ are $(a+h, k)$ and $4a$, respectively.
- The equation of the directrix of parabola $(y-k)^2 = 4a(x-h)$ is $x + a - h = 0$
- Parametric equations of parabola $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$.

Various Forms of the Equation of Chord

1. Point form

The equation of chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ is $y(y_1 + y_2) = 4ax + y_1y_2$.

2. Parametric form

The equation of chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$y(t_1 + t_2) = 2(x + at_1t_2)$.

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- Length of the chord $y = mx + c$ to the parabola $y^2 = 4ax$ is $\frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$.
- The length of the chord joining two points t_1 and t_2 on the parabola $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1+t_2)^2 + 4}$.

Condition for the Chord to be a Focal Chord

The chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is focal chord iff $t_1t_2 = -1$.

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- The length of the focal chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $a(t_2 - t_1)^2$.
- The length of the focal chord passing through the point t on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$.
- If one extremity of a focal chord is $(at_1^2, 2at_1)$, then the other extremity $(at_2^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ by virtue of relation $t_1t_2 = -1$.
- The focal chord of parabola $y^2 = 4ax$ making an angle α with the X-axis is of length $4a\,\text{cosec}^2\alpha$ and perpendicular on it from the vertex is $a \sin\alpha$.
- The length of a focal chord of a parabola varies inversely as the square of its distance from the vertex.
- If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latusrectum is $\frac{4l_1l_2}{l_1 + l_2}$.

Various Form of the Equation of Tangent

1. Point Form

The equation of tangents to all standard forms of parabola at point (x_1, y_1) are as given below

Equations of parabola	Equations of tangent
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

2. Parametric Form

The parametric equations of tangents to all standard forms of parabola are as given below

Equations of the parabola	Points of contact	Equations of tangent at t
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

3. Slope Form

The equation of tangents and condition of tangency of a line in terms of slope (m) are as given below

Parabola	Equations of tangent in terms of slope (m)	Points of contact	Condition of tangency of line $y = mx + c$
$y^2 = 4ax$	$y = mx + \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$c = \frac{a}{m}$
$y^2 = -4ax$	$y = mx - \left(\frac{a}{m}\right)$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$y = mx - am^2$	$(2am, am^2)$	$c = -am^2$
$x^2 = -4ay$	$y = mx + am^2$	$(-2am, -am^2)$	$c = am^2$

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- The tangent at any point P on the parabola bisects the angle between the focal chord through P and the perpendicular from P on the directrix.
- The portion of a tangent to a parabola cut-off between the directrix and the curve subtends a right angle at the focus.
- Tangents at the extremities of any focal chord intersect at right angle on the directrix.
- Any tangent to a parabola and the perpendicular on it from the focus, meet on the tangent at the vertex.
- Circle circumscribing the triangle formed by any tangent, normal and X-axis, has its centre at focus.

Various Form of the Equation of Normal

1. Point Form

The equation of normals to all standard forms of parabola at point (x_1, y_1) are as given below

Equations of parabola	Equations of normal
$y^2 = 4ax$	$y - y_1 = \frac{-y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

2. Parametric Form

The parametric equations of normals to all standard forms of parabola are as given below

Equations of parabola	Parametric coordinates	Equations of normals at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

3. Slope Form

The equation of normals and condition of normality of a line in terms of slope (m), are as given below

Equations of parabola	Point of contact in terms of slope (m)	Equations of normal in terms of slope (m)	Condition of normality of line $y = mx + c$
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

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- The line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, if $al(l^2 + 2m^2) + m^2 n = 0$.
- If the normals at points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meets parabola $y^2 = 4ax$ at the some point then $t_1 t_2 = 2$.
- If the normal at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.
- If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis, then it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.
- If the normal at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve. Then, the product of the ordinate of P and Q is $8a^2$.

Conormal Points

Any three points on the parabola at which the normals pass through a common point are called conormal points and these points are also known as the feets of the normals.

If three normals are drawn through a common point say (h, k) to the parabola $y^2 = 4ax$, then their slopes are the roots of the cubic equations $k = mh - 2am - am^3$.

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- The sum of the slopes of the normals at conormal points is zero, i.e. $m_1 + m_2 + m_3 = 0$.
- The sum of the ordinates of the conormal points is zero. For the parabola $y^2 = 4ax$, we have $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$.
- The centroid of the triangle formed by the conormal points lies on the axis of the parabola. If the vertices of the triangle formed by the conormal points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$, then y-coordinate of the centroid becomes $\frac{-2a(m_1 + m_2 + m_3)}{3} = \frac{-2a}{3} \times 0 = 0$. Hence, the centroid lies on the X-axis, i.e. axis of the parabola.
- If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) are real, then $h > 2a$.

Position of Points with Respect to Parabola

Let equation of parabola be $S \equiv y^2 - 4ax = 0$.

If value of parabola at $P(x_1, y_1)$ is $S_1 \equiv y_1^2 - 4ax_1$, then the following cases arise, which are as

(a) If $S_1 = 0$, then point lies on the parabola.
(b) If $S_1 < 0$, then point lies inside the parabola.
(c) If $S_1 > 0$, then point lies outside the parabola.

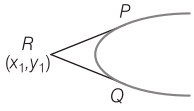
Number of Tangents Drawn from a Point to a Parabola

Two tangents can be drawn from a point to a parabola. The values of tangents are different, according to the position of a point, which are given as

(a) If a point lies outside, then two tangents are real and distinct.
(b) If a point lies inside, then two tangents are imaginary.
(c) If a point lies on the parabola, then two tangents are coincident.

Equation of Pair of Tangents

The equation of pair of tangents drawn from a point $R(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$



where, $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.

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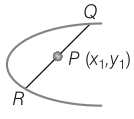
- Points of intersection of tangents at two points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R\{at_1t_2, a(t_1 + t_2)\}$ (where, x-coordinate of R is GM of x-coordinates of P, Q and y-coordinate of R is AM of y-coordinates of P, Q.)
- Angle θ between tangents at two points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $\tan\theta = \left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$.
- Locus of the point of intersection of perpendicular tangents to the parabola is directrix.

Equation of Chord Bisected at a Given Point

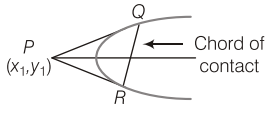
If any point $P(x_1, y_1)$ bisects the chord QR of the parabola $y^2 = 4ax$, then the equation of chord QR is, $T = S_1$



where, $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$.

Equation of Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$



where, $T = yy_1 - 2a(x + x_1)$.

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- The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.
- If tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, then the length of their chord of contact is $\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$.
- The area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$.

Diameter

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

Director Circle

Equation of director circle of all parabolas are given below

Equations of parabola	Equations of director circle
$y^2 = 4ax$	$x + a = 0$
$y^2 = -4ax$	$x - a = 0$
$x^2 = 4ay$	$y + a = 0$
$x^2 = -4ay$	$y - a = 0$

Note In case of parabola, director circle is directrix of the parabola.

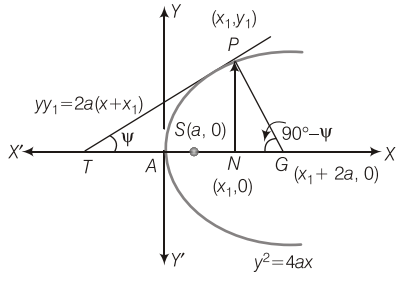
Pole and Polar

The polar of a point (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$ or $T = 0$.

Length of Tangent, Subtangent, Normal and Subnormal

Let the tangent and normal at any point $P(x_1, y_1)$ on a parabola $y^2 = 4ax$ meet the axis in T and G respectively, then PT = length of the tangent at P

PG = length of the normal at P



NT = length of subtangent at P

NG = length of subnormal at P

It can be seen from right angled ΔNPT and ΔPNG, that

(i) The length of tangent

$$= PT = PN \operatorname{cosec}\psi = y_1\sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$$

(ii) The length of the normal

$$= PG = PN\sec\psi = y_1\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}}$$

(iii) The length of subtangent

$$= NT = PN\cot\psi = y_1\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

(iv) The length of the subnormal

$$= NG = PN\tan\psi = y_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

where, $\tan\psi = \frac{2a}{y_1} = m$

Best Practice SHOTS

17. The equation of the parabola whose focus is the point (0, 0) and the tangent at the vertex is $x - y + 1 = 0$, is **[Odisha JEE 2002]**

(a) $x^2 + y^2 - 2xy - 4x + 4y - 4 = 0$
(b) $x^2 + y^2 - 2xy + 4x - 4y - 4 = 0$
(c) $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$
(d) $x^2 + y^2 + 2xy - 4x - 4y + 4 = 0$

18. The axis of the parabola $9y^2 - 16x - 12y - 57 = 0$, is

(a) $3y = 2$ (b) $x + 3y = 2$ (c) $2x = 3$ (d) $y = 3$

19. The equation of the directrix of the parabola whose vertex and focus are (1, 4) and (2, 6) respectively, is

(a) $x + 2y = 4$ (b) $x - y = 3$
(c) $2x + y = 5$ (d) $x + 3y = 8$

20. If the axis of the parabola $y^2 = 4ax$, its directrix and the chord joining t_1 and t_2 are concurrent, then t_1t_2 is equal to

(a) 0 (b) 1 (c) -1 (d) 2

21. If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then

(a) $2a^2 = bc$ (b) $a^3 = b^2c$ (c) $ac = b^2$ (d) $b^2c = 4a^3$

22. If the line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$, then **[Delhi CEE 1998]**

(a) $c = a + \frac{a}{m}$ (b) $c = am + \frac{a}{m}$
(c) $c = am + a$ (d) None of these

23. The line $y = mx + 1$ is a tangent to the parabola $y^2 = 4ax$, if m is equal to **[Delhi CEE 2000]**

(a) 1 (b) 2 (c) 4 (d) 3

24. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is **[JEE Main 2014]**

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{8}$ (d) $\frac{2}{3}$

25. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by **[IIT JEE 2011]**

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$
(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

26. If three distinct normals are drawn from $(2k, 0)$ to the parabola $y^2 = 4x$ such that one of them is X-axis and other two are perpendicular, then k is equal to

(a) 1 (b) $\frac{1}{2}$
(c) $\frac{3}{2}$ (d) None of these

27. If $(a^2, a - 2)$ is a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, – 4), then the set of all possible real values of a is

(a) $(-2, \sqrt{2})$ (b) $(-3, 2)$
(c) $(-2, 2\sqrt{2})$ (d) $(-2, -2 + \sqrt{2})$

28. The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$, is **[IIT JEE 2004]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

29. The locus of the mid-points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola, is

(a) $y^2 - 2ax + 2a^2 = 0$ (b) $y^2 - 2ax + 8a^2 = 0$
(c) $y^2 + 2ax - 8a^2 = 0$ (d) $y^2 - 2ax - 8a^2 = 0$

30. If the tangents are drawn at the points where the line $lx + my + n = 0$ is intersected by the parabola $y^2 = 4ax$, then the coordinates of the point of intersection of tangents are

(a) $\left(\frac{n}{l}, \frac{-2am}{l}\right)$ (b) $\left(\frac{l}{n}, -\frac{2am}{n}\right)$
(c) $\left(\frac{n}{m}, \frac{-2al}{m}\right)$ (d) None of these

31. The length of the chord of contact of tangents drawn from (x_1, y_1) to the parabola $y^2 = 4ax$ is

(a) $\frac{1}{a}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$.
(b) $\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$
(c) $\frac{1}{a}\sqrt{(y_1^2 + 4ax_1)(y_1^2 + 4a^2)}$
(d) None of the above

ELLIPSE

Ellipse as a Locus

An ellipse is the locus of a point which moves in a plane in such a way that its distance from a fixed point (i.e. focus) to a fixed line (i.e. directrix) is a constant ratio. This constant ratio is called eccentricity (e), which is less than 1, i.e. $0 < e < 1$.

Equation of an Ellipse in Standard Form

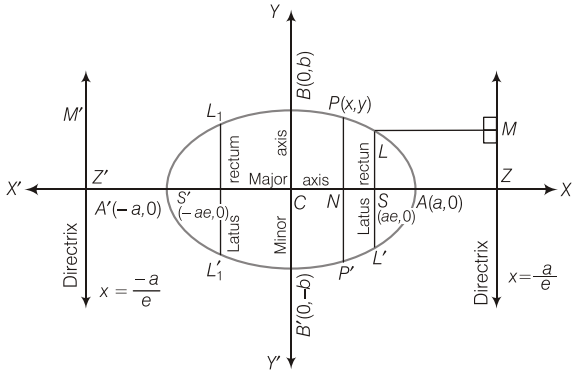
Standard form of the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$$

where, a and b are some constants and $b^2 = a^2(1 - e^2)$.

Some Important Terms and Properties of an Ellipse

The sketch of the locus of a moving point which satisfy the equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (where $a > b$) is shown in the following figure.



Some Important Terms

1. **Symmetry** On replacing y by $(-y)$ and x by $(-x)$, the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, remains same. So, the curve is symmetrical about X and Y-axes.
2. **Foci** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of an ellipse and the distance between the foci is given as $SS' = 2ae$.
3. **Directrices** The lines (ZM and $Z'M'$) are the two directrices of the ellipse and these equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively. Also, the distance between them is given as $ZZ' = \frac{2a}{e}$.
4. **Axes** The lines AA' and BB' are major and minor axes of an ellipse, respectively.
   (a) The length of major axis = $AA' = 2a$
   (b) The length of minor axis = $BB' = 2b$
5. **Centre** The point of intersection of the axes of an ellipse is known as the centre (C) of an ellipse.

Note All chords passing through C are bisected at C.

6. **Vertices** The end points A and A' of major axis are known as the vertices of an ellipse.
   The coordinates of A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

Note The vertex divides internally and externally, the join of focus and point of intersection of directrix with axis in the ratio of $e : 1$.

7. **Focal chord** A chord of an ellipse passing through its focus is called a focal chord.
8. **Ordinate and double ordinate** Let P be a point on the ellipse. From P, draw $PN \perp AA'$, then PN is called ordinate and PNP' is called double ordinate of the point P.
9. **Latusrectum** The focal chord perpendicular to the major axis is called latusrectum.
   If LL' and L_1L_1' are the latusrectum of an ellipse, then these lines are perpendicular to AA' and passing through S and S', respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right)$$

$$L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

$$L_1 \equiv \left(-ae, \frac{b^2}{a}\right)$$

$$L_1' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

   and length of latusrectum

$$= LL' = \frac{2b^2}{a} = L_1L_1'.$$

10. **Focal distance** The distance of any point on the ellipse from the focus is called focal distance.
    By definition of an ellipse,

$$\frac{SP}{PM} = e$$

$$\Rightarrow \quad SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$$

and $$S'P = e\left(\frac{a}{e} + x\right) = a + ex$$

Thus, the focal distances of any point $P(x, y)$ on the ellipse are $(a - ex)$ and $(a + ex)$.

Note $SP + S'P = 2a$, i.e. the sum of the distances of any point $P(x, y)$ on the ellipse from foci is always constant.

11. **Eccentricity of an ellipse** Since, $SP = ePM$

$\therefore (SP)^2 = e^2 (PM)^2$

$\Rightarrow (x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$

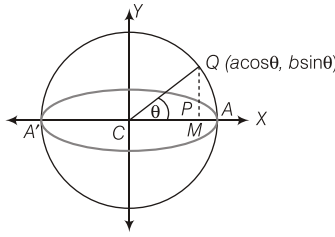
$\Rightarrow (x - ae)^2 + y^2 = (a - xe)^2$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$ ...(i)

On comparing Eq. (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get $b^2 = a^2(1 - e^2)$ or $e = \sqrt{1 - \frac{b^2}{a^2}}$.

12. **Auxiliary circle** A circle drawn by taking major axis (AA') as diameter is called auxiliary circle.

If equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of its auxiliary circle is $x^2 + y^2 = a^2$.



Let Q be any point on the auxiliary circle such that $QM \perp AA'$ cutting the ellipse at point P. The points P and Q are known as corresponding point on the ellipse and auxiliary circle, respectively.

Here, $\angle MCQ = \theta (0 \le \theta < 2\pi)$ is known as eccentric angle of any point P on the ellipse.

Note CQ but not CP is inclined at θ with X-axis.

Parametric Equation of an Ellipse

The parametric equations of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $x = a\cos\theta$ and $y = b\sin\theta$, where θ is a parameter and $0 < \theta < 2\pi$.

Equation of a Chord

Equation of chord joining two points P and Q with eccentric angles θ_1 and θ_2, respectively is

$$\frac{x}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)$$

Note (i) Equation of focal chord through $(ae, 0)$ is

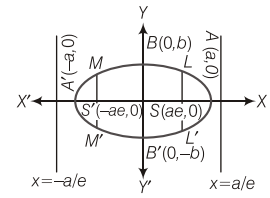
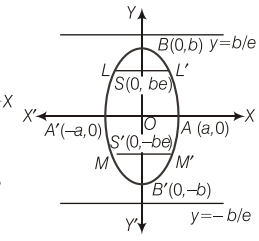
$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right) = \frac{e - 1}{e + 1}.$$

(ii) Equation of focal chord through $(-ae, 0)$ is

$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right) = \frac{e + 1}{e - 1}.$$

Standard Forms of Ellipse

Two standard forms of the ellipse are shown below along with their properties

Standard equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ (Horizontal form of an ellipse)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a < b)$ (Vertical form of an ellipse)
Shape of the ellipse		
Centre	(0, 0)	(0, 0)
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 - b^2}{a^2}}$	$e = \sqrt{\frac{b^2 - a^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latusrectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$
Focal distances of point (x, y)	$SP = a - ex$ and $S'P = a + ex$	$SP = b - ey$ and $S'P = b + ey$
Sum of focal distances $(SP + S'P)$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$\frac{2a}{e}$	$\frac{2b}{e}$
Tangents at the vertices	$x = \pm a$	$y = \pm b$

Various Form of Equation of Tangent to an Ellipse

1. Point Form

The equation of tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

Note The equation of tangent at (x_1, y_1) to any second degree curve can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{xy_1+x_1y}{2}$ and without changing the constant (if any) in the equation of curve.

2. Parametric Form

The equation of tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point $(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

3. Slope Form

The equation of tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with slope m is $y=mx\pm\sqrt{a^2m^2+b^2}$.

Condition of Tangency and Point of Contact

The line $y=mx+c$ is said to be tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ iff $c^2=a^2m^2+b^2$. The coordinates of point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

Various Form of Equation of Normal to an Ellipse

1. Point Form

The equation of the normal at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

2. Parametric Form

The equation of the normal at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(ax\sec\theta-by\,\mathrm{cosec}\,\theta)=a^2-b^2$.

3. Slope Form

The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope m is $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$.

Condition of Normality and Point of Contact

The line $y=mx+c$ is normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ iff $c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}$

The coordinates of the point of contact are

$$\left[\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right]$$

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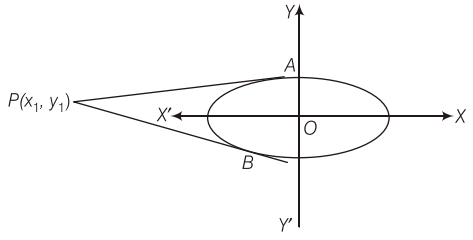
- In general four normals can be drawn to an ellipse from a point in its plane.
- If there are four points on an ellipse at which the normals pass through a common point. These four points are called the conormal points.
- If α, β, γ and δ are the eccentric angles of these four conormal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.
- If α, β and γ are the eccentric angles of these three conormal points, then $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$.

Equation of Pair of Tangents

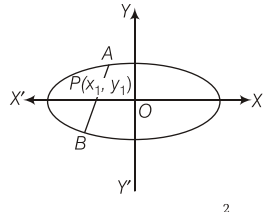
The equation of the pair of tangents drawn from any point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $SS_1=T^2$



where, $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$; $S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

Equation of Chord Bisected at a Given Point

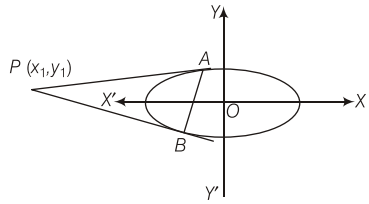
If a point $P(x_1, y_1)$ bisects the chord AB of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of chord AB is given by $T=S_1$



where, $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ and $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$.

Equation of Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $T = 0$



where, $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

Diameter

The equation of the diameter bisecting chords of slope m of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $y = -\frac{b^2}{a^2 m}x$.

Conjugate Diameters

Two diameters of an ellipse are said to be conjugate diameters, if each bisects the chords parallel to the other. Two straight lines $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $m_1 m_2 = \frac{-b^2}{a^2}$.

Director Circle

The equation of director circle of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$.

Note The product of perpendiculars from the foci on any tangent to the ellipse is b^2.

Pole and Polar

The polar of a point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

Best Practice SHOTS

32. The equation of an ellipse whose axes are along the coordinate axes with vertices $(\pm 5, 0)$ and foci $(\pm 4, 0)$ is

(a) $9x^2 + 25y^2 = 1$ (b) $9x^2 + 25y^2 = 225$
(c) $25x^2 + 9y^2 = 225$ (d) $25x^2 + 9y^2 = 1$

33. The length of the axes of the conic $9x^2 + 4y^2 - 6x + 4y + 1 = 0$ are **[Odisha JEE 2002]**

(a) $\frac{1}{2}, 9$ (b) $3, \frac{2}{5}$ (c) $1, \frac{2}{3}$ (d) $3, 2$

34. The eccentricity of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$, is **[Delhi CEE 2009]**

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

35. The equation of the ellipse whose axes are along the coordinate axes, with vertices $(0, \pm 10)$ and eccentricity $e = \frac{4}{5}$, is

(a) $36x^2 + 100y^2 = 3600$ (b) $36x^2 + 100y^2 = 1$
(c) $100x^2 + 36y^2 = 3600$ (d) $100x^2 + 36y^2 = 1$

36. If any two chords are drawn through two points on the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ equidistant from the centre. If α, β, γ and δ are the eccentric angles of the extremities of the chords, then the value of $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}$, is

(a) -1 (b) 1 (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

37. The locus of the middle point of the portion of a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ included between the axis, is the curve

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$ (b) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$
(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 4$ (d) None of these

38. If an ellipse slides between two perpendicular lines, then the locus of its centre is

(a) a parabola (b) an ellipse
(c) a pair of straight lines (d) a circle

39. The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at positive end of the latusrectum is

(a) $x + ey + e^3 a = 0$
(b) $x - ey - e^2 a = 0$
(c) $x - ey - e^3 a = 0$
(d) None of the above

40. Tangents are drawn from the points $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B. The coordinates of A and B are **[IIT JEE 2010]**

(a) (3, 0) and (0, 2) (b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$
(c) $\left(\frac{-8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2) (d) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

{JEE FINAL TOUCH}

41. The locus of mid-points of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a}$ (b) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{ex}{a}$

(c) $x^2+y^2=a^2+b^2$ (d) None of these

42. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to

(a) $\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$

(c) $-\frac{a^4}{b^4}$ (d) $-\frac{b^4}{a^4}$

43. If AB is a diameter of $x^2+9y^2=25$ and the eccentric angle of A is $\frac{\pi}{6}$, then the eccentric angle of B is

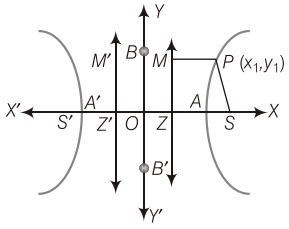
(a) $\frac{5\pi}{6}$ (b) $\frac{-5\pi}{6}$

(c) $-\frac{2\pi}{3}$ (d) None of these

HYPERBOLA

Hyperbola as a Locus

A hyperbola is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point (i.e. focus) and a fixed line (i.e. directrix) is always constant, which is greater than one. This ratio is called eccentricity (e).



Let S and MZ be the focus and directrix of the hyperbola.

If $P(x_1, y_1)$ is any point on the hyperbola, then

$$\frac{PS}{PM}=e \text{ or } PS=ePM,\ e>1$$

or

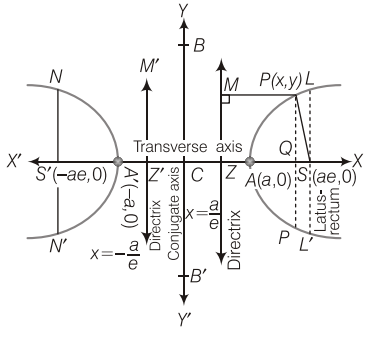
A hyperbola is the locus of a point which moves in such a way that the difference of its distance from two fixed points (i.e. foci) is always constant.

Standard Equation of Hyperbola

Standard equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

where, a and b are constants and $b^2=a^2(e^2-1)$

The sketch of the locus of a moving point which satisfy the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is shown in the following figure.



Some Important Terms

1. **Symmetry** On replacing y by – y and x by – x, then the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ remains same. So, the curve is symmetrical about both X and Y-axes.
2. **Foci** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola and the distance between foci is $SS'=2ae$.
3. **Directrices** The lines ZM and $Z'M'$ are called the directrices of the hyperbola and its equations are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$. Also, distance between the directrices is $ZZ'=\frac{2a}{e}$.
4. **Axes** The lines AA' and BB' are known as transverse axis and conjugate axis, respectively of the hyperbola.
   The length of the transverse axis = $AA'=2a$
   The length of the conjugate axis = $BB'=2b$
5. **Centre** The point of intersection of the axes of the hyperbola is known as centre (C) of the hyperbola.

Note All chords passing through C are bisected at C.

6. **Vertices** The end points A and A' of transverse axis are known as the vertices of a hyperbola.
   The coordinates of A and A' are $(a, 0)$ and $(-a, 0)$, respectively.
7. **Focal chord** A chord of a hyperbola passing through its focus is known as focal chord.
8. **Focal distance** The distance of any point on the hyperbola from the focus is called focal distance.
   The difference of the focal distances of any point on the hyperbola is always constant and this constant is always equal to the length of the transverse axis of the hyperbola.
   If P is any point of the hyperbola, then $S'P-SP=2a$
   (= length of transverse axis)
9. **Latusrectum** The focal chord perpendicular to the transverse axis is called latusrectum.

If LL' and NN' are the latusrectum of a hyperbola, then these lines are perpendicular to the transverse axis AA' and passing through S and S', respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), L' \equiv \left(ae, \frac{-b^2}{a}\right), N \equiv \left(-ae, \frac{b^2}{a}\right)$$

$$N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

and the length of latusrectum $= LL' = \frac{2b^2}{a} = NN'$

10. **Ordinate and double ordinate** Let P be a point on the hyperbola. From P, draw PQ perpendicular to the axis of the hyperbola and produced to meet the curve again at P'. Then, PQ is called ordinate and PP' is called double ordinate.
11. **Eccentricity of hyperbola** The eccentricity of a hyperbola is $e = \sqrt{1 + b^2 / a^2}$.
12. **Auxiliary circle** Equation of auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$.
13. **Parametric equations of the hyperbola** The parametric equations of the hyperbola are $x = a\sec\theta$ and $y = b\tan\theta$, where θ is the parameter and $0 < \theta < 2\pi$.
14. **Equation of chord** The equation of chord joining the points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ is

$$\begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & b\tan\theta_1 & 1 \\ a\sec\theta_2 & b\tan\theta_2 & 1 \end{vmatrix} = 0$$

or $\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$.

Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola, is called the conjugate hyperbola of the given hyperbola. The conjugate hyperbola of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ i.e. } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

Properties of hyperbola and conjugate are given below in the table

	Hyperbola	Conjugate hyperbola
Standard equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0, 0)	(0, 0)
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{a^2 + b^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parameter coordinates	$(a\sec\theta, b\tan\theta)$	$(a\tan\theta, b\sec\theta)$
Focal distances of point (x_1, y_1)	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal distances $(S'P - SP)$	$2a$	$2b$
Tangents at the vertices	$x = \pm a$	$y = \pm b$

Position of a Point with Respect to a Hyperbola

Let an equation of the hyperbola be $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$

If value of a hyperbola at (x_1, y_1) is $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$, then the following cases arise, which are as

(a) If $S_1 = 0$, then the point lies on the hyperbola.
(b) If $S_1 < 0$, then the point lies inside the hyperbola.
(c) If $S_1 > 0$, then the point lies outside the hyperbola.

Various Form of the Equation of Tangent

1. Point Form

If $P(x_1, y_1)$ is any point on the hyperbola, then the equation of tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

2. Parametric Form

If $P(a\sec\theta, b\tan\theta)$ is any point on the hyperbola, then the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

3. Slope Form

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

Condition for Tangency and Points of Contact

A line $y = mx + c$ is said to be tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ iff $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm \frac{a^2 m}{\sqrt{a^2 m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2 m^2 - b^2}} \right).$$

Nature of Tangents Drawn from a Point to a Hyperbola

Two tangents can be drawn from a point to a hyperbola. The nature of tangents depends on the position of a point, which are given as

(i) If a point lies outside, then two tangents are real and distinct.

(ii) If a point lies on the hyperbola, then two tangents are coincident.

(iii) If a point lies inside the hyperbola, then two tangents are imaginary.

Various Form of the Equation of Normal

1. Point Form

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$.

2. Parametric Form

The equations of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(a\sec\theta, b\tan\theta)$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

3. Slope Form

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope m is $y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$

The coordinates of points of contact are

$$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2 m^2}}, \mp \frac{mb^2}{\sqrt{a^2 - b^2 m^2}} \right)$$

Note Four normals can be drawn from any point to a hyperbola.

Conormal Points

Any four points on the hyperbola at which the normals pass through a common point are called conormal points.

Note Tangent drawn at any point bisects the angle between the lines joining the point to the foci whereas normal bisects the supplementary angle between the lines.

Equation of Pair of Tangents

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $SS_1 = T^2$,

where $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Equation of Chord Bisected at a Given Point

If $P(x_1, y_1)$ bisects the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

then the equation of chord bisected at the point P is $T = S_1$,

where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

and $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$.

Equation of Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$,

where $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$.

Diameter

The equation of a diameter bisecting a system of parallel chord of slope m of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2 m}x$.

Conjugate Diameter

Two diameters of a hyperbola are said to be conjugate diameters, if each bisects the chords parallel to the other.

Two lines, $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $m_1 m_2 = \frac{b^2}{a^2}$.

Director Circle

The equation of director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

Pole and Polar

The polar of a point (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$.

Asymptotes

If the length of perpendicular drawn from a point on the hyperbola to a straight line tends to zero as the point moves to infinity, then the straight line is known as asymptotes.

The equations of two asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ are } y = \pm \frac{b}{a}x \text{ or } \frac{x}{a} \pm \frac{y}{b} = 0.$$

Rectangular Hyperbola

A hyperbola whose asymptotes are perpendicular to each other is called rectangular hyperbola.

or

If the length of transverse and conjugate axes of any hyperbola is equal, then it is called rectangular hyperbola.

Various Form of Rectangular Hyperbola

1. Cartesian Form

(a) The general equation of the rectangular hyperbola is

$$x^2 - y^2 = a^2$$

(b) If we take coordinate axes along the asymptotes of a rectangular hyperbola, then the equation of rectangular hyperbola becomes $xy = c^2$, where c is any constant.

2. Parametric Form

The parametric equations of rectangular hyperbola $xy = c^2$ are $x = ct$ and $y = \frac{c}{t}$, where t is any parameter.

Properties of Rectangular Hyperbola $x^2 - y^2 = a^2$

1. The equations of asymptotes of the rectangular hyperbola are $y = \pm x$.
2. The transverse and conjugate axes of a rectangular hyperbola are equal in length.
3. Eccentricity $(e) = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}$

Properties of Rectangular Hyperbola $xy = c^2$

1. Equation of the chord joining t_1 and t_2 is $x + yt_1t_2 - c(t_1 + t_2) = 0$.
2. Equation of tangent at (x_1, y_1) is $xy_1 + x_1y = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$.
3. Equation of tangent at $(ct, c/t)$ to the hyperbola is $\frac{x}{t} + yt = 2c$.
4. Point of intersection of tangents at $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.
5. Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.
6. Equation of normal at t is $xt^3 - yt - ct^4 + c = 0$.
7. The equation of the chord of the hyperbola $xy = c^2$, whose middle point is (x_1, y_1), is $T = S_1$, i.e. $xy_1 + x_1y = 2x_1y_1$.
8. The slope of the tangent at the point $(ct, c/t)$ is $-1/t^2$, which is always negative. Hence, tangents drawn at any point to $xy = c^2$ would always make an obtuse angle with the X-axis.
9. The slope of the normal at the point $(ct, c/t)$ is t^2, which is always positive. Hence, normals drawn to $xy = c^2$ at any point would always make an acute angle with the X-axis.

Continued from Page 17 (Formulae at a Glance)

11. Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

12. The foot of the perpendicular (h, k) from (x_1, y_1) to the line $ax + by + c = 0$ is given by

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}.$$

13. If the lines represented by $ax^2 + 2hxy + by^2 = 0$ are $y - m_1x = 0$ and $y - m_2x = 0$, then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

14. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

15. Angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ or $ax^2 + 2hxy + by^2 = 0$ is given by $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

Lines are parallel, if $h^2 = ab$ and $af^2 = bg^2$ or $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

16. Equation of bisector of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Best Practice SHOTS

44. The equation of the hyperbola, whose foci are (8, 3), (0, 3) and eccentricity $= \frac{4}{3}$, is

(a) $49x^2 - 3y^2 = 147$
(b) $7x^2 - 9y^2 = 63$
(c) $7(x-4)^2 - 9(y-3)^2 = 63$
(d) $7(x+4)^2 - 9(y+3)^2 = 63$

45. The eccentricity of the hyperbola
$9x^2 - 16y^2 + 72x - 32y - 16 = 0$, is

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) $\frac{9}{16}$ (d) $\frac{16}{9}$

46. If e and e' are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e'^2}$ is equal to [Delhi CEE 2006]

(a) 0 (b) 1
(c) 2 (d) None of these

47. The line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

(a) $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$ (b) $a^2\cos^2\alpha - b^2\sin^2\alpha = p$
(c) $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ (d) $a^2\cos^2\alpha + b^2\sin^2\alpha = p$

48. The equations of the tangents to the hyperbola $3x^2 - y^2 = 3$, which are perpendicular to the line $x + 3y = 2$, are

(a) $y = 3x \pm \sqrt{6}$ (b) $y = 3x \pm 2\sqrt{3}$
(c) $y = 3x \pm \sqrt{3}$ (d) None of these

49. If the normal at 'θ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at G and A, A' are the vertices of the hyperbola, then $AG \cdot A'G$ is equal to

(a) $a^2(e^2\sec^2\theta - 1)$ (b) $a^2(e^4\sec^2\theta - 1)$
(c) $a^2(e^4\sec^2\theta + 1)$ (d) None of these

50. The coordinates of the point of intersection of tangents drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the points, where it is intersected by the line $lx + my + n = 0$, are

(a) $\left(\frac{a^2l}{n}, \frac{-b^2m}{n}\right)$ (b) $\left(\frac{a^2l}{n}, \frac{b^2m}{n}\right)$
(c) $\left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$ (d) $\left(-\frac{a^2l}{n}, \frac{-b^2m}{n}\right)$

51. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is

(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
(c) $4x - 4y = 3$ (d) $3x - 4y = 2$

52. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. The area cut-off by the chord of contact on the asymptotes is equal to

(a) $\frac{ab}{2}$ (b) ab (c) $2ab$ (d) $4ab$

53. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is

(a) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$
(b) $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$
(c) $2x^2 + 5xy + 2y^2 = 0$
(d) None of the above

54. If the tangent and normal to a rectangular hyperbola cut-off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other axis, then

(a) $a_1a_2 = b_1b_2$ (b) $a_1b_2 = a_2b_1$
(c) $a_1a_2 + b_1b_2 = 0$ (d) $a_1b_2 + a_2b_1 = 0$

Answers

1. (c)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (d)
11. (c)	**12.** (b)	**13.** (b)	**14.** (d)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** (b)
21. (d)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (a,b,d)	**26.** (c)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (a)
31. (a)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (d)	**39.** (c)	**40.** (d)
41. (a)	**42.** (c)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (b)	**47.** (a)	**48.** (a)	**49.** (b)	**50.** (c)
51. (a)	**52.** (d)	**53.** (a)	**54.** (c)						

Formulae at a Glance

1. General equation of circle is $x^2+y^2+2gx+2fy+c=0$, whose radius $=\sqrt{g^2+f^2-c}$ and centre $=(-g,-f)$

2. Different equations of circle are

(i) $(x-h)^2+(y-k)^2=r^2$

(ii) $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

(when end points of diameter are given)

3. The lengths of intercepts made by the circle $x^2+y^2+2gx+2fy+c=0$ on X and Y-axes are $2\sqrt{g^2-c}$ and $2\sqrt{f^2-c}$, respectively.

4. *(i)* Equation of tangent at (x_1, y_1) to circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$.

(ii) Equation of tangent at (x_1, y_1) to circle $x^2+y^2+2gx+2fy+c=0$ is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

(iii) Equation of tangent at $(a\cos\theta, a\sin\theta)$ to circle $x^2+y^2=a^2$ is $x\cos\theta+y\sin\theta=a$.

5. Line $y=mx+c$ is a tangent to the circle $x^2+y^2=a^2$, if $c^2=a^2(1+m^2)$ and point of contact is $\left(\frac{\pm ma}{\sqrt{1+m^2}}, \frac{\mp a}{\sqrt{1+m^2}}\right)$.

6. The length of intercept cut-off from the line $y=mx+c$ by the circle $x^2+y^2=a^2$ is $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$.

7. Length of tangent from external point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}=\sqrt{S_1}$.

8. *(i)* Equation of pair of tangent is $SS_1=T^2$,

where $S=x^2+y^2+2gx+2fy+c$

$S_1=x_1^2+y_1^2+2gx_1+2fy_1+c$

and $T=xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$

(ii) Angle between the two tangents from (α,β) to the circle $x^2+y^2=r^2$ is $2\tan^{-1}\frac{r}{\sqrt{S_1}}$.

9. If a line is normal to the circle, then it passes through the centre of circle.

10. Equation of chord of contact from outside point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

If $P(x_1, y_1)$ is the mid-point of chord, then equation of chord is $T=S_1$, i.e.

$xx_1+yy_1+g(x+x_1)+f(y+y_1)=x_1^2+y_1^2+2gx_1+2fy_1$

11. Equation of common chord of two circles is $S_1-S_2=0$.

12. Equation of director circle $x^2+y^2=r^2$ is $x^2+y^2=2r^2$.

13. Two circles cut orthogonally, if $2g_1g_2+2f_1f_2=c_1+c_2$

14. *(i)* Equation of circle passing through the point of intersection of line (L) and circle (S) is $S+\lambda L=0$.

(ii) Equation of circle passing through the points of intersections of circles $S_1=0$ and $S_2=0$ is $S_1+\lambda S_2=0$.

15. The second degree equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$, represents

(i) a parabola, if $\Delta\neq 0, ab-h^2=0, e=1$

(ii) an ellipse, if $\Delta\neq 0, ab-h^2>0, e<1$

(iii) a hyperbola, if $\Delta\neq 0, ab-h^2<0, e>1$

(iv) a rectangular hyperbola, if $\Delta\neq 0$, $ab-h^2<0$, $a+b=0$, $e=\sqrt{2}$; where $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

16.

Important terms	Parabola $y^2=4ax$	Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Centre	(0, 0)	(0, 0)	(0, 0)
Focus/Foci	$(a, 0)$	$(\pm ae, 0)$	$(\pm ae, 0)$
Vertex/vertices	(0, 0)	$(\pm a, 0)$	$(\pm a, 0)$
Equation of directrix/directrices	$x+a=0$	$x=\pm\frac{a}{e}$	$x=\pm\frac{a}{e}$
Length of latusrectum	$4a$	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Ends of latusrectum	$(a, \pm 2a)$	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm ae, \pm\frac{b^2}{a}\right)$
Parametric equation	$x=at^2, y=2at$	$x=a\cos\alpha, y=b\sin\alpha$	$x=a\sec\alpha, y=b\tan\alpha$
Distance between foci	-	$2ae$	$2ae$
Equation of tangent at point (x_1, y_1)	$yy_1=2a(x+x_1)$	$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$	$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$
Equation of tangent in slope form	$y=mx+\frac{a}{m}$	$y=mx\pm\sqrt{a^2m^2+b^2}$	$y=mx\pm\sqrt{a^2m^2-b^2}$
Equation of tangent in parametric form	$ty=x+at^2$	$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$	$\frac{x}{a}\sec\alpha-\frac{y}{b}\tan\alpha=1$

Important terms	Parabola $y^2 = 4ax$	Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Line $y = mx + c$ touches, if	$c = \frac{a}{m}$	$c^2 = a^2m^2 + b^2$	$c^2 = a^2m^2 - b^2$
Point of contact of $y = mx + c$ and curve	$\left(\frac{a}{m^2}, \frac{2a}{m}\right), m \neq 0$	$\left(\frac{\pm a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$	$\left(\frac{\pm a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 - b^2}}\right)$
Equation of normal at point (x_1, y_1)	$y - y_1 = \frac{-y_1}{2a}(x - x_1)$	$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$	$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$
Equation of normal in parametric form	$y + tx = 2at + at^3$	$ax\sec\alpha - by\,\mathrm{cosec}\,\alpha = a^2 - b^2$	$ax\cos\alpha - by\cot\alpha = a^2 + b^2$
Equation of normal in slope form	$y = mx - 2am - am^3$	$y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$	$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$
Condition for normality	$c = -2am - am^3$	$c = \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$	$c = \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - m^2b^2}}$
Equation of pair of tangents is $SS_1 = T^2$	$S = y^2 - 4ax$ $S_1 = y_1^2 - 4ax_1$ $T = yy_1 - 2a(x + x_1)$	$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$	$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$
Equation of chord of contact	$yy_1 = 2a(x + x_1)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
Equation of chord with mid-point (x_1, y_1)	$T = S_1$	$T = S_1$	$T = S_1$
Equation of diameter	$y = \frac{2a}{m}$	$y = -\frac{b^2}{a^2m}x$	$y = \frac{b^2}{a^2m}x$
Equation of polar	$yy_1 = 2a(x + x_1)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
Coordinates of pole (line $lx + my + n = 0$)	$\left(\frac{n}{l}, -\frac{2am}{l}\right)$	$\left(-\frac{a^2l}{n}, -\frac{b^2m}{n}\right)$	$\left(-\frac{a^2l}{n}, \frac{b^2m}{n}\right)$
Auxiliary circle	—	$x^2 + y^2 = a^2$	$x^2 + y^2 = a^2$
Director circle	—	$x^2 + y^2 = a^2 + b^2$	$x^2 + y^2 = a^2 - b^2$
Eccentricity	$e = 1$	$e = \sqrt{\left(\frac{a^2 - b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2 + b^2}{a^2}\right)}$

17. (i) Point $P(x_1, y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =, < 0$.

(ii) The point of intersection of tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $(at_1t_2, a(t_1 + t_2))$.

(iii) If the normal at $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

(iv) Equation of chord joining $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ is $y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$.

If it passes through the focus, then

$$t_2 = -\frac{1}{t_1}$$

$$\Rightarrow \quad t_1t_2 = -1$$

18. Point $P(x_1, y_1)$ lies outside, on or inside the ellipse according as $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 >, =, < 0$

19. (i) Point (x_1, y_1) lies outside, on or inside the hyperbola according as $<, =, > 0$.

(ii) The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$ or $\frac{x}{a} \pm \frac{y}{b} = 0$.

20. (i) For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(\pm c\sqrt{2}, \pm c\sqrt{2})$ and directrices are $x + y = \pm c\sqrt{2}$.

(ii) Equation of tangent to the hyperbola $xy = c^2$ at (x_1, y_1) is $xy_1 + yx_1 = 2c^2$ and at $\left(ct, \frac{c}{t}\right)$ is $\frac{x}{t} + yt = 2c$.

(iii) Equation of normal to the hyperbola $xy = c^2$ at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$ and at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - yt - ct^4 + c = 0$.

Mathematical Reasoning

Reasoning is the main asset that makes humans superior than the other species. The ability of reasoning varies from person to person. In the context of Mathematics, there are two kinds of reasoning, *viz*. inductive and deductive reasoning. Inductive reasoning starts with specific examples or observations and deduces the apparent rules or patterns that lie behind them, whereas deductive reasoning starts with the rules and determines what the consequences will be. This is what we do in most of Maths, defining the rules for a mathematical entity and using it to prove that other facts are true.

With the development of mathematical reasoning, students recognise that the Mathematics make sense and can be understood. It is an important topic for JEE Main. One question is always asked from this topic. For convenience, we divide this topic into two parts, *viz*. 'Statements' and 'Logical equivalence'.

STATEMENTS OR PROPOSITIONS

In day-to-day life, we convey our thoughts in the form of sentences and these sentences are of the following types

(i) Declarative/Assertive sentence

(ii) Imperative sentence

(iii) Exclamatory sentence

(iv) Interrogative sentence

Now, let us look at the following definitions

Statements or Propositions

A declarative sentence, whose truth or falsity can be determined is called a statement.

Note that statements cannot be define as imperative, exclamatory and interrogative sentence, because truth or falsity of these sentences cannot be determine. Also, a statement cannot be both true and false at the same time.

(a) **Valid statement** If a statement is true, then it is called valid statement.

(b) **Invalid statement** If a statement is false, then it is called invalid statement.

Logical Variables

In the study of logic, statements are represented by lower case letters such as $p, q, r, s,\ldots$ etc. These letters are called logical variables. e.g. The statement, New Delhi is a city, may be represented or denoted by p and we write p : New Delhi is a city.

Truth Value

The truth or falsity of a statement is called its truth value. If a statement is true, then we say that its truth value is 'True' or '*T*'. On the other hand, if the statement is false, then we say that its truth value is 'False' or '*F* '.

Quantifiers

The symbol $\forall$ (stands for 'for all') and $\exists$ (stands for 'there exists') are known as quantifiers.

Types of Statements

1. **Simple statement** A statement, which cannot be broken into two or more statements, is called a simple statement.

   In other words, any statement whose truth value does not explicity depend on another statement, is said to be a simple statement.

2. **Compound statement** A statement formed by two or more simple statements by the words such as 'and', 'or', 'not', 'if then', 'if and only if', then the resulting statement is called a compound statement.

   Here, the phrases or words which connect simple statements are called logical connectives or sequential connectives or simply connectives or logical operators.

   In the following table, we list some possible connectives, their symbols and the nature of the compound statement formed by them.

Connective	Symbol	Nature of the compound statement formed by using the connective
and	$\wedge$	**Conjunction** If any two simple statements are combined by the word 'and', then the resulting compound statement is called the conjunction of the original statements. Symbolically, if p and q are two simple statements, then $p \wedge q$ denotes the conjunction of p and q.
or	$\vee$	**Disjunction or alternation** If any two simple statements are combined by the word 'or' then the resulting compound statement is called the disjunction of the original statements. Symbolically, if p and q are two simple statements, then $p \vee q$ denotes the disjunction of p and q.

Connective	Symbol	Nature of the compound statement formed by using the connective
if……then	$\Rightarrow$ or $\rightarrow$	**Implication or conditional** If any two simple statements are connected by the phrase 'if … then', then the resulting compound statement is called implication or conditional statement. If p and q are two statements forming the implication 'if p then q', then we denote this implication by '$p \Rightarrow q$' or '$p \rightarrow q$'.
if and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	**Equivalence or bi-conditional** A statement is a bi-conditional statement, if it is the conjunction of two conditional statements one converse to the other. In other words, if p and q are two statements connected by the connective, 'if and only if', then the resulting compound statement 'p if and only if q' is called bi-conditional statement and written as '$p \Leftrightarrow q$' or '$p \leftrightarrow q$'.
not	$\sim$ or $\neg$	**Negation** The denial of a statement p is called its negation, written as $\sim p$. Negation of any statement p is formed by writing, 'it is not the case that … or it is false that', before p or, if possible by inserting the word 'not' in p.

Note 'Not' is called a connective although it does not combine two or more statements. In facts, it only modifies a statement.

SPEED BOOSTER

If p and q are two statements, then here is a quick definition of their converse, inverse and contrapositive.

- The converse of $p \Rightarrow q$ is $q \Rightarrow p$
- The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$
- The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

3. **Sub-statements** Simple statements which when combined to form a compound statement are called sub-statements, also called components or constituents.
4. **Open statement** A declarative sentence containing one or more variables, such that when certain values are given to the variable it becomes a statement, is called an open statement. e.g. Statement, p : x is an integer; is an open statement.
   Truth set The set of all those values of the variable(s), for which open statement becomes a true statement, is called the truth set of the open statements, e.g. For the open statement, p : x is an integer, truth set = Z = the set of integers.
5. **Quantified statement** The statements containing quantifiers are known as quantified statement. e.g. $2x^2 > 0, \forall x \in Z$ is a quantified statement.

Truth Table

A table that shows the relationship between the truth value of a compound statement $S(p, q, r...)$ and the truth value of its sub-statements $p, q, r...$ etc, is called the truth table of statement S.

Construction of Truth Table

In order to construct the truth table for a compound statement, we follow the following steps

***Step* I** Prepare a table consisting of rows and columns.

***Step* II** At the top of the columns, write the variables denoting the sub-statements or consistent statements. Also, write compound statement in symbolic form.

***Step* III** Write the truth value of each sub-statements.

***Step* IV** Write the truth value of compound statement on the basis of truth values of the sub-statements.

Note If a compound statement is made up of two simple statements, then the number of rows in the truth table will be 2^2 and if it is made up of three simple statements, then the number of rows will be 2^3. In general, if it is made up of n simple statements, then its truth table will contain 2^n rows.

Now, let us look at the following truth tables

1. The truth table for operation 'and' is given by

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. The truth table for operation 'or' is given by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3. The truth table for operation 'if … then' is given by

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

4. **Truth table for a bi-conditional statement** Since, $p \Leftrightarrow q$ is the conjunction of $p \Rightarrow q$ and $q \Rightarrow p$. So, we have the following truth table for $p \Leftrightarrow q$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$ or $(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

5. Truth table for operation 'not' is given by

p	$\sim p$
T	F
F	T

Best Practice SHOTS

1. Which of the following is a proposition?
(a) I am an advocate
(b) A half open door is half closed
(c) Delhi is on the Jupiter
(d) None of the above

2. Which of the following is not a proposition?
(a) 3 is a prime (b) $\sqrt{2}$ is irrational
(c) Mathematics is interesting (d) 5 is an even integer

3. Consider the following propositions
p : I take medicine. q : I can sleep.
Then, the compound statement $\sim p \rightarrow \sim q$ means
(a) if I do not take medicine, then I cannot sleep.
(b) if I do not take medicine, then I can sleep
(c) I take medicine iff I can sleep
(d) I take medicine, if I can sleep

4. Consider the following propositions
p : It rains; q : The street gets flooded.
The proposition, 'if it does not rain, then the street does not get flooded' is represented by
(a) $p \rightarrow \sim q$ (b) $\sim p \rightarrow q$
(c) $p \leftrightarrow q$ (d) $\sim p \rightarrow \sim q$

5. Consider the following propositions
p : To become an air force officer one should be graduate.
q : To become an air force officer one should have good health.
The compound proposition 'To become an air force officer one should be graduate and should have good health' is represented by
(a) $p \vee q$ (b) $p \rightarrow q$
(c) $p \wedge q$ (d) $p \leftrightarrow q$

6. Let p and q be two propositions given by
p : I play cricket during the holidays.
q : I just sleep throughout the day.
Then, the compound statement $p \vee q$ is
(a) If I play cricket during the holidays, then I just sleep throughout the day
(b) I play cricket during the holidays and just sleep throughout the day
(c) I just sleep throughout the day if and only if I play cricket during the holidays
(d) I play cricket during the holidays or I just sleep throughout the day

7. Let p : 7 is not greater than 4 and q: Paris is in France, be two statements. Then the statement $\sim(p \vee q)$ is
(a) 7 is greater than 4 or Paris is not in France
(b) 7 is not greater than 4 and Paris is not in France
(c) 7 is not greater than 4 and Paris is in France
(d) 7 is greater than 4 and Paris is not in France

8. If p : It is snowing; q: I am cold, then the compound statement 'It is snowing and it is not that I am cold' is given by **[Kerala PET 2012]**
(a) $p \wedge (\sim q)$ (b) $p \wedge q$
(c) $(\sim p) \wedge q$ (d) $(\sim p) \wedge (\sim q)$

9. Which of the following is not a negation of the statement $p : \sqrt{5}$ is rational?
(a) It is not the case that $\sqrt{5}$ is rational.
(b) $\sqrt{5}$ is not rational
(c) $\sqrt{5}$ is an irrational
(d) None of the above

10. Contrapositive of p : 'If x and y are integers such that xy is odd then both x and y are odd' is
(a) if both x and y are odd, then xy is odd
(b) if both x and y are even, then xy is even
(c) if x or y is odd, then xy is odd
(d) if it is false that both x and y are odd, then the product xy is odd

11. The contrapositive of $(p \vee q) \Rightarrow r$ is **[KCET 1999]**
(a) $r \Rightarrow (p \vee q)$ (b) $\sim r \Rightarrow (p \vee q)$
(c) $\sim r \Rightarrow \sim p \wedge \sim q$ (d) $p \Rightarrow (q \vee r)$

12. Which of the following is open statement?
(a) x is a natural number (b) Give me a glass of water
(c) Wish you best of luck (d) Good morning to all

13. Let S be a non-empty subset of R. Consider the following statement
p : There is a rational number $x \in S$ such that $x > 0$.
Which of the following statement is the negation of the statement p? **[AIEEE 2010]**
(a) There is a rational number $x \in S$ such that $x \leq 0$.
(b) There is no rational number $x \in S$ such that $x \leq 0$
(c) Every rational number $x \in S$ satisfies $x \leq 0$
(d) $x \in S$ and $x \leq 0 \Rightarrow x$ is not rational

14. If $x = 5$ and $y = -2$, then $x - 2y = 9$. The contrapositive of this proposition is
(a) if $x - 2y \neq 9$, then $x \neq 5$ or $y \neq -2$
(b) if $x - 2y = 9$, $x \neq 5$ and $y \neq -2$
(c) $x - 2y = 9$ if and only if $x = 5$ and $y = -2$
(d) None of the above

15. The contrapositive of $2x + 3 = 9 \Rightarrow x \neq 4$ is
(a) $x = 4 \Rightarrow 2x + 3 \neq 9$ (b) $x = 4 \Rightarrow 2x + 3 = 9$
(c) $x \neq 4 \Rightarrow 2x + 3 \neq 9$ (d) $x \neq 4 \Rightarrow 2x + 3 = 9$

16. If the inverse of implication $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$, then the inverse of the proposition $(p \wedge \sim q) \rightarrow r$ is
(a) $\sim r \rightarrow \sim p \vee q$ (b) $\sim p \vee q \rightarrow \sim r$
(c) $r \rightarrow p \wedge \sim q$ (d) None of these

17. The inverse of the proposition $(\sim p \wedge q) \vee \sim r \to s$ is
(a) $\sim p \wedge \sim q \vee \sim r \to \sim s$
(b) $(p \vee \sim q) \wedge r \to \sim s$
(c) $\sim p \vee \sim q \wedge \sim r \to \sim s$
(d) None of the above

18. Converse of the statement,
If a number n is even, then n^2 is even, is
(a) if a number n^2 is even, then n is even
(b) if n is not even, then n^2 is not even
(c) neither n nor n^2 is even
(d) None of the above

19. The compound statement $p \to (\sim p \vee q)$ is false, then the truth values of p and q are respectively
(a) T, T (b) T, F (c) F, T (d) F, F

20. If p, q and r are simple proposition, then $(p \wedge q) \wedge (q \wedge r)$ is true, then
(a) p, q, r are all false (b) p, q, r are all true
(c) p, q are true and r is false (d) p is true and q and r are false

21. If p, q and r are simple proposition with truth values T, F, T, then the truth value of $(\sim p \vee q) \wedge \sim r \Rightarrow p$ is
(a) true (b) false
(c) true, if r is false (d) true, if q is true

LOGICALLY EQUIVALENT STATEMENT

Equivalent Statement

Two compounds statements $S_1(p, q, r, \ldots)$ and $S_2(p, q, r, \ldots)$ are said to be logically equivalent or simply equivalent, if they have the same truth values for all logically possibilities.

It follows from the above definition that two statements S_1 and S_2 are logically equivalent, if they have identical truth table i.e. the entries in the last column of the truth table are same.

If statements $S_1(p, q, r, \ldots)$ and $S_2(p, q, r, \ldots)$ are logically equivalent, then we write $S_1(p, q, r, \ldots) \equiv S_2(p, q, r, \ldots)$

Negation of Compound Statement

Following are the negation of different compound statements

(i) **Negation of conjunction** If p and q are two statements, then $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

(ii) **Negation of disjunction** If p and q are two statements, then $\sim(p \vee q) \equiv (\sim p \wedge \sim q)$

(iii) **Negation of implication** If p and q are two statements, then $\sim(p \Rightarrow q) \equiv (p \wedge \sim q)$

(iv) **Negation of bi-conditional statement or equivalence** If p and q are two statements, then $\sim(p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Tautologies and Contradictions

Let $p, q, r, \ldots$ be statements, then any statement involving $p, q, r, \ldots$ and the logical connectives $\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$ is called a statement pattern. e.g.

(i) $p \vee q$
(ii) $p \Rightarrow q$
(iii) $((p \wedge q) \vee r) \Rightarrow (s \wedge \sim s)$
(iv) $(p \Rightarrow q) \Leftrightarrow (p \Rightarrow \sim p)$ etc. are statement patterns.

In other words, a compound statement with the repetitive use of the logical connectives is called a statement pattern or a well-formed formula.

Tautology A statement pattern is called a tautology, if it is always true, whatever may be the truth values of its component statements.

A tautology is called a theorem or a logically valid statement pattern. A tautology contains only T in the last column of its truth table.

Contradiction A statement pattern is called a contradiction, if it is always false, whatever may be the truth values of its component statements. In the last column of the truth table of contradiction, there is always F.

The negation of a tautology is a contradiction and *vice-versa*.

Contingency

A compound statement which is neither a tautology nor a contradiction is called a contingency.

Algebra of Statements

The following are some laws of algebra of statements

(i) **Idempotent law** If p is a statement, then
(a) $p \vee p \equiv p$ (b) $p \wedge p \equiv p$

(ii) **Commutative law** If p and q are statements, then
(a) $p \vee q \equiv q \vee p$ (b) $p \wedge q \equiv q \wedge p$

(iii) **Associative law** If p, q and r are statements, then
(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(iv) **Distributive law** For any three statements p, q and r, we have
(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(v) **De-Morgan's law** If p and q are two statements, then
(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ (b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(vi) **Identity law** If t and c denote a tautology and a contradiction respectively, then for any statement p, we have
(a) $p \wedge t \equiv p$ (b) $p \vee c \equiv p$ (c) $p \vee t \equiv t$ (d) $p \wedge c \equiv c$

(vii) **Complement law** For any statement p, we have
(a) $p \vee \sim p = t$ (b) $p \wedge \sim p = c$ (c) $\sim t = c$ (d) $\sim c = t$
where, t and c denote a tautology and a contradiction respectively.

(viii) **Law of contrapositive** For any two statements p and q, we have $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(ix) **Involution law** For any statement p, we have $\sim(\sim p) \equiv p$

Duality

Two compound statements S_1 and S_2 are said to be duals of each other, if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. Now, let us look at some important facts

(i) The connectives $\wedge$ and $\vee$ are also called duals of each other.

(ii) If a compound statement contains the special variable t (tautology) or c (contradiction), then to obtain its dual, we replace t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

(iii) Let $S(p, q)$ be a compound statement containing two sub-statements and $S^*(p, q)$ be its dual. Then,

(a) $\sim S(p, q) \equiv S^*(\sim p, \sim q)$ (b) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

(iv) The above results can be extended to the compound statement having finite number of sub-statements. Thus, if $S(p_1, p_2, \ldots, p_n)$ is a compound statement containing n sub-statements $p_1, p_2, \ldots, p_n$ and $S^*(p_1, p_2, \ldots, p_n)$ is its dual. Then,

(a) $\sim S(p_1, p_2, \ldots, p_n) \equiv S^*(\sim p_1, \sim p_2, \ldots, \sim p_n)$

(b) $\sim S^*(p_1, p_2, \ldots, p_n) \equiv S(\sim p_1, \sim p_2, \ldots, \sim p_n)$

Best Practice SHOTS

22. Which of the following statement is not equivalent to $p \leftrightarrow q$?

(a) p if and only if q
(b) q if and only if p
(c) p is necessary and sufficient for q
(d) None of the above

23. The statement $p \rightarrow (q \vee r)$ is not equivalent to

(a) $(p \rightarrow q) \vee (p \rightarrow r)$ (b) $p \wedge (\sim q) \rightarrow r$
(c) $p \wedge (\sim r) \rightarrow q$ (d) $p \wedge q \rightarrow (p \wedge r) \vee (q \wedge r)$

24. Which of the following is not equivalent to $\sim p \wedge q$?

(a) $\sim (q \rightarrow p)$ (b) $\sim (p \vee \sim q)$
(c) $\sim [\sim p \rightarrow \sim q]$ (d) $\sim (p \vee q)$

25. Which of the following is equivalent to $p \leftrightarrow q$?

(a) $(\sim p \vee q) \vee (p \vee q)$ (b) $(p \wedge q) \vee (\sim p \wedge \sim q)$
(c) $(p \vee q) \wedge (p \vee \sim q)$ (d) $(p \wedge q) \vee (p \vee q)$

26. $\sim (p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to

[Kerala (Engg.) 2010]

(a) $\sim p$ (b) p (c) q (d) $\sim q$

27. $\sim q \wedge p$ is logically equivalent to

(a) $p \rightarrow q$ (b) $q \rightarrow p$
(c) $\sim (p \rightarrow q)$ (d) $\sim (q \rightarrow p)$

28. Given that water freezes below zero degree celsius. Consider the following statements.

p : Water froze this morning.

q : This morning temperature was below 0°C.

Which of the following is correct?

(a) p and q are logically equivalent
(b) p is the inverse of q
(c) p is the converse of q
(d) p is the contrapositive of q

29. The logically equivalent proposition of $p \Leftrightarrow q$ is

[KCET 2000]

(a) $(p \wedge q) \vee (p \wedge q)$ (b) $(p \Rightarrow q) \wedge (q \Rightarrow p)$
(c) $(p \wedge q) \vee (q \Rightarrow p)$ (d) $(p \wedge q) \Rightarrow (q \vee p)$

30. Negation of 'Ram is in class X or Rashmi is in class XII' is

(a) Ram is not in class X but Ram is in class XII
(b) Ram is not in class X but Rashmi is not in class XII
(c) either Ram is not in class X or Ram is not in class XII
(d) None of the above

31. The negation of the compound proposition $p \vee (\sim p \vee q)$ is

(a) $(p \wedge \sim q) \wedge \sim p$ (b) $(p \wedge \sim q) \vee \sim p$
(c) $(p \vee \sim q) \vee \sim p$ (d) None of these

32. Consider the following statements

P : Suman is brilliant.

Q : Suman is rich.

R : Suman is honest.

The negation of the statement, 'Suman is brilliant and dishonest if and only if Suman is rich' can be expressed as [AIEEE 2011]

(a) $\sim P \wedge (Q \leftrightarrow \sim R)$ (b) $\sim (Q \leftrightarrow (P \wedge \sim R))$
(c) $\sim Q \leftrightarrow \sim P \wedge R$ (d) $\sim (P \wedge \sim R) \leftrightarrow Q$

33. The negation of the proposition 'If a quadrilateral is a square, then it is a rhombus' is

(a) if a quadrilateral is not a square, then it is a rhombus
(b) if a quadrilateral is a square, then it is not a rhombus.
(c) a quadrilateral is a square and it is not a rhombus
(d) a quadrilateral is not a square and it is a rhombus.

34. Which of the following is always true?

(a) $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$
(b) $\sim (p \vee q) \equiv (\sim p \vee \sim q)$
(c) $\sim (p \rightarrow q) \equiv (p \vee \sim q)$
(d) $\sim (p \wedge q) \equiv (\sim p \wedge \sim q)$

35. The negation of $p \rightarrow (\sim p \vee q)$ is

(a) $p \vee (p \vee \sim q)$ (b) $p \rightarrow \sim (p \vee q)$
(c) $p \rightarrow q$ (d) $p \wedge \sim q$

36. The negation of $p \rightarrow (\sim p \vee \sim q)$ is

(a) $p \vee (p \vee \sim q)$ (b) $p \rightarrow \sim (p \vee q)$
(c) $p \rightarrow q$ (d) $p \wedge q$

37. The proposition $p \rightarrow \sim(p \wedge \sim q)$ is

(a) a contradiction
(b) a tautology
(c) either a tautology or a contradiction
(d) neither a tautology nor a contradiction

38. $\sim(p \Rightarrow q) \Leftrightarrow \sim p \vee \sim q$ is

(a) a tautology
(b) a contradiction
(c) neither a tautology nor a contradiction
(d) Cannot come to any conclusion

39. The conditional $(p \wedge q) \Rightarrow p$ is

(a) a tautology
(b) a fallacy, i.e. contradiction
(c) neither tautology nor fallacy
(d) None of the above

40. The only statement among the following that is a tautology, is [AIEEE 2011]

(a) $A \wedge (A \vee B)$ (b) $A \vee (A \wedge B)$
(c) $[A \wedge (A \rightarrow B)] \rightarrow B$ (d) $B \rightarrow [A \wedge (A \rightarrow B)]$

41. Which one of the following is a tautology?

(a) $[(p \rightarrow q) \wedge p] \rightarrow q$ (b) $[(p \rightarrow q) \vee p] \rightarrow q$
(c) $[(p \rightarrow q) \vee \sim p] \rightarrow q$ (d) $[(p \rightarrow q) \wedge (\sim q)] \rightarrow p$

42. If p always speaks against q, then $p \Rightarrow p \vee \sim q$ is

(a) a tautology (b) contradiction
(c) contingency (d) None of these

43. Let p, q and r be three statements, then $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$ is a

(a) tautology (b) contradiction
(c) fallacy (d) None of these

44. The false statement in the following is [KCET 2002]

(a) $(p \wedge (\sim p))$ is a contradiction
(b) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction
(c) $\sim(\sim p) \Leftrightarrow p$ is a tautology
(d) $p \vee (\sim p)$ is a tautology

45. Which of the following is a contradiction?

(a) $(p \wedge q) \wedge \sim(p \vee q)$ (b) $p \vee (\sim p \wedge q)$
(c) $(p \Rightarrow q) \Rightarrow p$ (d) None of these

46. If a compound statement r is contradiction, then the truth value of $(p \Rightarrow q) \wedge r \wedge [p \Rightarrow \sim r]$ is

(a) T (b) F
(c) T or F (d) None of these

47. Which of the following is contingency?

(a) $p \vee \sim p$ (b) $p \wedge q \Rightarrow p \vee q$
(c) $p \wedge \sim q$ (d) None of these

48. The property $\sim(p \wedge q) \equiv \sim p \vee \sim q$ is called

(a) associative law (b) De-Morgan's law
(c) commutative law (d) idempotent law

49. Dual of $(p \rightarrow q) \rightarrow r$ is

(a) $(q \rightarrow p) \wedge r$ (b) $p \rightarrow (q \rightarrow r)$
(c) $(p \vee \sim q) \vee r$ (d) None of these

50. If $S(p, q, r) = (\sim p) \vee [\sim(q \wedge r)]$ is a compound statement, then $S(\sim p, \sim q, \sim r)$ is [Kerala (Engg.) 2010]

(a) $\sim S(p, q, r)$ (b) $S(p, q, r)$
(c) $p \vee (q \wedge r)$ (d) $p \vee (q \vee r)$

Answers

1. (c)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (a)	**9.** (d)	**10.** (d)
11. (c)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (b)	**20.** (b)
21. (a)	**22.** (d)	**23.** (d)	**24.** (d)	**25.** (b)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (d)
31. (a)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (d)	**36.** (d)	**37.** (d)	**38.** (c)	**39.** (a)	**40.** (c)
41. (a)	**42.** (a)	**43.** (a)	**44.** (b)	**45.** (a)	**46.** (b)	**47.** (c)	**48.** (b)	**49.** (a)	**50.** (d)

Formulae at a Glance

1. Table for basic logical connections

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

2. (i) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv q \leftrightarrow p$
(ii) $\sim(p \rightarrow q) \equiv p \wedge \sim q$
(iii) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

3. (i) $p \vee p = p$ (ii) $p \wedge p = p$

4. (i) $p \vee q = q \vee p$ (ii) $p \wedge q = q \wedge p$

5. (i) $(p \vee q) \vee r = p \vee (q \vee r)$ (ii) $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

6. (i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
(ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

7. (i) $\sim(p \wedge q) = \sim p \vee \sim q$ (ii) $\sim(p \vee q) = \sim p \wedge \sim q$

8. (i) $p \wedge T = p$ (ii) $p \vee T = T$
(iii) $p \vee F = p$ (iv) $p \wedge F = F$

9. (i) $p \wedge \sim p = F$ (ii) $p \vee \sim p = T$

10. (i) Contrapositive law $p \rightarrow q \equiv \sim q \rightarrow \sim p$
(ii) Converse law $p \rightarrow q \equiv q \rightarrow p$
(iii) Inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$

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Questions to Measure Your Problem Solving Skills

1. Two finite sets A and B have p and q elements, respectively. If the total number of subsets of A is 192 more than the total number of subsets of B, then the value of q is

(a) 8 (b) 6 (c) 10 (d) 9

2. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, then $|w| = 1$, implies that in the complex plane

(a) z lies on the real axis (b) z lies on the imaginary axis
(c) z lies on the unit circle (d) None of these

3. If 1, ω and ω^2 are cube roots of unity, then $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ ... to $2n$ factors is equal to

(a) 2^n (b) 2^{2n} (c) 0 (d) 1

4. If A is a 3-rowed square matrix such that $|A| = 2$, then $|\text{adj } \{\text{adj}(\text{adj } A^2)\}|$ is equal

(a) 2^4 (b) 8
(c) 2^{16} (d) None of these

5. The equations

$$9x_1 - 3x_2 = \lambda x_1 + 9;\ -3x_1 + 12x_2 - 3x_3 = \lambda x_2 + 12;$$
$$-3x_2 + 9x_3 = \lambda x_3 + 9$$

have solutions for $\lambda \in S$, where S equals

(a) {9, 12} (b) {9, 15, 6}
(c) {9, 6} (d) None of these

6. How many odd numbers, greater than 600000 can be formed from the digits 5, 6, 7, 8, 9, 0, where repetition is not allowed?

(a) 240 (b) 480
(c) 360 (d) None of these

7. If the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000, then the value of x is

(a) 10 (b) 100
(c) 1 (d) None of these

8. The harmonic mean of two numbers is 4, their AM, A and GM, G satisfy the relation, $2A + G^2 = 27$, the numbers are

(a) 1, 3 (b) 1, 4
(c) 3, 6 (d) None of these

9. The sum to n terms of the series

$$\frac{3}{1\cdot 2}\left(\frac{1}{2}\right) + \frac{4}{2\cdot 3}\left(\frac{1}{2}\right)^2 + \frac{5}{3\cdot 4}\left(\frac{1}{2}\right)^3 + \ldots \text{ is}$$

(a) $1 - \frac{1}{(n+1)2^n}$ (b) $1 - \frac{1}{(n+1)2^{n-1}}$
(c) $1 - \frac{1}{(n-1)2^{n-1}}$ (d) None of these

10. $\lim\limits_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ is equal to

(a) 0 (b) 1 (c) 1/2 (d) None

11. Let $f(x) = \begin{cases} \frac{1}{|x|}; & |x| \geq 1 \\ ax^2 + b; & |x| < 1 \end{cases}$, if $f(x)$ is continuous and differentiable at any point, then

(a) $a = \frac{1}{2}, b = \frac{-3}{2}$ (b) $a = -\frac{1}{2}, b = \frac{3}{2}$
(c) $a = 1, b = -1$ (d) None of these

12. The equation of the normal to the curve $y = e^{-2|x|}$ at the point, where the curve cuts the line $x = \frac{1}{2}$, is

(a) $2e\,(ex + 2y) = e^2 - 4$ (b) $2e\,(ex - 2y) = e^2 - 4$
(c) $2e\,(ey - 2x) = e^2 - 4$ (d) None of these

13. If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then value of $\lim\limits_{x\to a} \frac{g(x)f(a) - g(a)f(x)}{x - a}$ is

(a) 5 (b) $\frac{1}{5}$
(c) –5 (d) None of these

14. $\int \frac{dx}{(x-1)^2\sqrt{1-x}}$ is equal to

(a) $\frac{1}{3}\sqrt{1+x}(2+x)(1-x)^{-3/2}$

(b) $\frac{1}{3}\sqrt{1-x}(2+x)(1+x)^{-3/2}$

(c) $\frac{1}{3}\sqrt{1+x}(2-x)(1-x)^{-3/2}$

(d) None of the above

15. The integral $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$ is equal to

(a) $\frac{\pi a^4}{16}$ (b) $\frac{3\pi a^4}{16}$ (c) $\frac{\pi a^4}{14}$ (d) $\frac{\pi a^4}{13}$

16. The area of the region bounded by the pairs of lines $y = |x-1|$ and $y = 3-|x|$ is

(a) 3 sq units (b) 4 sq units (c) 6 sq units (d) 2 sq units

17. The solution of the differential equation $\frac{dy}{dx} - 3y\cot x = \sin 2x$, where $y = 2$ when $x = \frac{\pi}{2}$, is

(a) $y = 2\cos^2 x + 4\cos^3 x$

(b) $y = -2\cos x + 4\cos^2 x$

(c) $y = -2\sin^2 x + 4\sin^3 x$

(d) $y = 2\sin^2 x - 4\sin^3 x$

18. The condition, that the pair of straight line joining the origin to the intersections of line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ is the right angles, is

(a) $2c^2 = a^2(1-m^2)$ (b) $2c^2 = a^2(1+m^2)$

(c) $c^2 = a^2(1+m^2)$ (d) None of these

19. Let P be the image of the point $(-3, 2)$ with respect to X-axis. Keeping the origin as same, the coordinate axes are rotated through an angle $60°$ in the clockwise sense. The coordinates of point P with respect to the new axes are

(a) $\left(\frac{2\sqrt{3}-3}{2}, -\frac{3\sqrt{3}+2}{2}\right)$ (b) $\left(\frac{2\sqrt{3}-3}{2}, \frac{3\sqrt{3}+2}{2}\right)$

(c) $\left(-\frac{2\sqrt{3}-3}{2}, \frac{3\sqrt{3}+2}{2}\right)$ (d) None of these

20. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is

(a) 0 (b) 1

(c) 3 (d) 4

21. If the tangents from the point $(\lambda, 3)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are at right angle, then the value of λ is

(a) ± 1 (b) ± 3

(c) ± 2 (d) ± 4

22. The parabola $y^2 = kx$ makes an intercept of length 4 on the line $x - 2y = 1$, then k is equal to

(a) $\frac{\sqrt{105}-5}{10}$ (b) $\frac{5-\sqrt{105}}{10}$

(c) $\frac{5+\sqrt{105}}{10}$ (d) None of these

23. The distance between the planes $x + 2y - 3z - 4 = 0$ and $2x + 4y - 6z = 1$ along the line $\frac{x}{1} = \frac{y}{-3} = \frac{z}{2}$ is

(a) $\frac{19}{22}$ (b) $\frac{3}{22}$

(c) 5 (d) None of these

24. A variable plane remains at a constant distance $3p$ from the origin and cuts the axes at A, B, C. The locus of centroid of ΔABC is

(a) $x^{-1} + y^{-1} + z^{-1} = p^{-1}$ (b) $x^{-2} + y^{-2} + z^{-2} = p^{-2}$

(c) $x^{-3} + y^{-3} + z^{-3} = p^{-9}$ (d) None of these

25. If the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are inclined at angle $2\theta (0 \le \theta \le \pi)$ and $|\hat{\mathbf{a}} - \hat{\mathbf{b}}| < 1$, then θ lies in the interval

(a) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (b) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$ (c) $\left[0, \frac{\pi}{2}\right]$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

26. A speaks truth in 75% and B in 80% of the cases. In what percentage of cases are they likely to contradict each other narrating the same incident?

(a) 5% (b) 45% (c) 35% (d) 15%

27. The mean of n items is $\bar{x}$. If the first term is increased by 1, second term by 2 and so on, then new mean is

(a) $\bar{x} + n$ (b) $\bar{x} + \frac{n}{2}$

(c) $\bar{x} + \frac{n+1}{2}$ (d) None of these

28. The flagstaff stands vertically on a pillar, the height of the flagstaff being double the height of the pillar. A man on the ground at a distance finds that both the pillar and the flagstaff subtend equal angles at his eyes. The ratio of the height of the pillar and distance of the man from the pillar is

(a) $\sqrt{3}:1$ (b) $1:3$ (c) $1:\sqrt{3}$ (d) $\sqrt{3}:2$

29. If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2$ is equal to

(a) $36\sin^2\theta$ (b) $36\cos^2\theta$ (c) 36 (d) $-36\sin^2\theta$

30. The negation of $p \to (\sim p \vee q)$ is

(a) $p \vee (p \vee \sim q)$ (b) $p \to \sim(p \wedge q)$

(c) $p \to q$ (d) $p \wedge (\sim q)$

Answers

1. (b)	**2.** (a)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (b)
11. (b)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (b)	**19.** (a)	**20.** (b)
21. (c)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (c)	**26.** (c)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (d)

BINOMIAL THEOREM

In Mathematics, the binomial theorem is used to expand binomial expression $(x + a)$ raised to any given power without direct multiplication.

BINOMIAL THEOREM FOR POSITIVE INTEGER

If n is any positive integer, then

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_na^n$$

or $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}a^r$

This is the binomial theorem for positive integer.

Here, ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients and

$${}^nC_r = \frac{n!}{r!(n-r)!}, \forall\ 0 \le r \le n$$

Some Useful Properties of Binomial Theorem for Positive Integer

1. Total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$.
2. The sum of indices of x and a in each term is n.
3. The expansion of $(x + a)^n$ is also true when x and a are complex numbers.
4. The binomial coefficient of terms equidistant from the beginning and the end are equal, i.e.
$${}^nC_r = {}^nC_{n-r}, \forall r = 0, 1, 2, \ldots, n$$
5. General term in the expansion of $(x + a)^n$ is $T_{r+1} = {}^nC_r x^{n-r}a^r$
6. The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.
7. Independent term or constant term of a binomial expansion is the term in which exponent of the variable (s) is zero.
8. (i) If n is odd, then $(x + a)^n + (x - a)^n$ and $(x + a)^n - (x - a)^n$ both have same number of terms and equal to $\left(\frac{n+1}{2}\right)$.

   (ii) If n is even, then $(x + a)^n + (x - a)^n$ has $\left(\frac{n+1}{2}\right)$ terms and $(x + a)^n - (x - a)^n$ has $\left(\frac{n}{2}\right)$ terms.
9. In binomial expansion of $(x + a)^n$, the rth term from the end is equal to $(n - r + 2)$th term from the beginning.

If n is any positive integers, then

- $(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + \ldots + (-1)^n\ {}^nC_na^n$,
  i.e. $(x-a)^n = \sum_{r=0}^{n}(-1)^r\ {}^nC_r \cdot x^{n-r}a^r$
- $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$
  i.e. $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$
- $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \ldots + (-1)^n\ {}^nC_nx^n$, i.e. $(1-x)^n = \sum_{r=0}^{n}(-1)^r\ {}^nC_r x^r$
- $(1+x)^n + (1-x)^n = 2\,[C_0 + C_2x^2 + C_4x^4 + \ldots]$
- $(1+x)^n - (1-x)^n = 2[C_1x + C_3x^3 + \ldots]$
- $(x+a)^n + (x-a)^n = 2({}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + \ldots)$
- $(x+a)^n - (x-a)^n = 2({}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + \ldots)$
- $C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1} = n(1+x)^{n-1}$
  $C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots + C_n\frac{x^{n+1}}{n+1} = \int_0^x (1+t)^n\,dt$

MIDDLE TERM

(i) If n is even, then middle term in the expansion of $(x + a)^n$ is $\left(\frac{n}{2}+1\right)$th term.

(ii) If n is odd, then middle term in the expansion of $(x + a)^n$ are $\left(\frac{n+1}{2}\right)$th term and $\left(\frac{n+3}{2}\right)$th term.

GREATEST COEFFICIENT OR GREATEST BINOMIAL COEFFICIENT

(i) If n is even, then the greatest coefficient in the expansion of $(x + a)^n$ is ${}^nC_{n/2}$.

(ii) If n is odd, then the greatest coefficient in the expansion of $(x + a)^n$ are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$ both being equal.

GREATEST TERM

In the expansion of $(x + a)^n$

(i) If $\left(\frac{n+1}{\left|\frac{x}{a}\right|+1}\right)$ is an integer $= p$ (say), then the greatest terms are T_p and T_{p+1} both being equal.

(ii) If $\left(\frac{n+1}{\left|\frac{x}{a}\right|+1}\right)$ is not an integer with m as an integral part of $\left(\frac{n+1}{\left|\frac{x}{a}\right|+1}\right)$, then T_{m+1} is the greatest term.

Some Useful Properties of Binomial Coefficients

If ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are the binomial coefficients in the expansion of $(x + a)^n$, then

1. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ 2. $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$

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MULTINOMIA

For any $n \in N$

(i) $(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1+r_2+r_3+\ldots+r_k=n} \frac{n!}{r_1!r_2!\ldots}$

(ii) The general term in the expansion of $(x_1 + x_2 + \ldots +$

(iii) The greatest coefficient in the expansion of $(x_1 + x_2 +$

where q and r are the quotient and remainder respecti

(iv) Number of distinct or dissimilar terms in the expans
non-negative integral solution of $r_1 + r_2 + \ldots + r_k =$

Note If n is any positive integer, then number of term

Similarly, number of terms in the expansion o

3. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
4. $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$
5. If $a_0, a_1, a_2, \ldots, a_n$ are in AP, then sum of the series
   $a_0C_0 + a_1C_1 + \ldots + a_nC_n = (a_0 + a_n)\,2^{n-1}$
6. $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^nC_n = 0$
7. $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$
8. For $n > 1$,
   $C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1}\,nC_n = 0$
9. $C_0 - C_2 + C_4 - C_6 + \ldots = (\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right)$
10. $C_1 - C_3 + C_5 - C_7 + \ldots = (\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right)$
11. $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^rC_r = (-1)^{r\ n-1}C_r, r < n$
12. $C_0C_r + C_1C_{r+1} + \ldots + C_{(n-r)}C_n = {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$

on through PT MAP

earn quickly...

AL THEOREM

$\frac{}{\ldots r_k!}\, x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

$x_k)^n$ is $\frac{n!}{r_1!\, r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

$+ \ldots + x_k)^n$ is $\frac{n!}{(q!)^{k-r}[(q+1)!]^r}$,

ively, when n is divisible by k.

sion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is equal to number of n, i.e. ${}^{n+k-1}C_{k-1}$

ns in the expansion of $(x + y + z)^n$ is $\frac{(n+1)(n+2)}{2}$.

of $(w + x + y + z)^n$ is $\frac{(n+1)(n+2)(n+3)}{6}$.

13. $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{(n!)^2}$

14. $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots$
$= \begin{cases} 0, \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, \text{if } n \text{ is even} \end{cases}$

15. $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{(n+1)} = \frac{2^{n+1}-1}{(n+1)}$

16. $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots + (-1)^n \frac{C_n}{(n+1)}$
$= \frac{1}{n+1}$

17. $C_0 + \frac{C_1}{2^1} + \frac{C_2}{2^2} + \frac{C_3}{2^3} + \ldots + \frac{C_n}{2^n} = \left(\frac{3}{2}\right)^n$

18. $\sum_{r=0}^{n} (-1)^r {}^nC_r \left\{ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots \text{ to } m \text{ terms} \right\} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$

DIVISIBILITY PROBLEMS

From the expansion of

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$,

we have following result which are given below.

(i) $(1+x)^n - 1 = {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$ is divisible by x, i.e. it is multiple of x or $(1+x)^n - 1 = M(x)$.

(ii) $(1+x)^n - 1 - nx = {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + {}^nC_n x^n = M(x^2)$

(iii) $(1+x)^n - 1 - nx - \frac{n(n-1)}{2} x^2 = {}^nC_3 x^3 + {}^nC_4 x^4 + \ldots + {}^nC_n x^n = M(x^3)$

R-f FACTOR RELATIONS

- If $(\sqrt{A}+B)^n = I + f$, where I and n are positive integers, n being odd and $0 \le f < 1$, then $(I+f)f = k^n$, where $k = A - B^2 > 0$ and $\sqrt{A} - B < 1$.
- If n is even integer, then $(\sqrt{A}+B)^n + (\sqrt{A}-B)^n = I + f + f'$

where $f' = (\sqrt{A}-B)^n = 1 - f$. In that case $(I+f)(1-f) = k^n$, where $k = A - B^2$

BINOMIAL THEOREM FOR ANY INDEX

Let n be a rational number and x be a real number, such that $|x| < 1$, then.

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \infty$

Note

(i) The condition $|x| < 1$ is unnecessary, if n is a whole number while the same condition is essential, if n is a rational number other than a whole number.

(ii) There are infinite number of terms in the expansion of $(1+x)^n$, when n is a rational number other than a whole number.

(iii) If n is a positive integer, then the above expansion contains $(n+1)$ terms and coincide with $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

EXPANSION OF $(x+a)^n$

To expand $(x+a)^n$, where n is a rational number other than a whole number, *we have following two cases*

Case I When $x > a$

Then, $\frac{a}{x} < 1$, in this case we have

$(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n = x^n\left(1+\frac{a}{x}\right)^n = x^n\left\{1 + n\cdot\frac{a}{x} + \frac{n(n-1)}{2!}\cdot\left(\frac{a}{x}\right)^2 + \ldots\right\}$

Case II When $x < a$

Then, $\frac{x}{a} < 1$, in this case we have

$(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left\{1 + n\cdot\frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \ldots\right\}$

SOME USEFUL TIPS AND TRICKS

1. To find remainder when a^n is divided by 'b' try to express 'a' or some of its powers as $kb \pm 1$ and then use binomial theorem, where k is some scalar, e.g., To find remainder when 7^{200} is divided by 50, we write $7^2 = 50 - 1$ and then use binomial theorem.
2. If the coefficient of rth, $(r+1)$th and $(r+2)$th terms of $(1+x)^n$ are in AP, then
$$n^2 - (4r+1)n + 4r^2 = 2$$
3. (a) The coefficient of x^{n-1} in the expansion of
$$(x-1)(x-2)\ldots(x-n) = -\frac{n(n+1)}{2}$$
(b) The coefficient of x^{n-1} in the expansion of
$$(x+1)(x+2)\ldots(x+n) = \frac{n(n+1)}{2}$$

GENERAL TERM

The general term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

SOME STANDARD EXPANSIONS

For $|x| < 1$,

(i) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$

(ii) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

(iii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$

MULTINOMIAL THEOREM

For any $n \in N$

(i) $(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1 + r_2 + r_3 + \ldots + r_k = n} \frac{n!}{r_1!\, r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

(ii) The general term in the expansion of $(x_1 + x_2 + \ldots + x_k)^n$ is $\frac{n!}{r_1!\, r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

(iii) The greatest coefficient in the expansion of $(x_1 + x_2 + \ldots + x_k)^n$ is $\frac{n!}{(q!)^{k-r}[(q+1)!]^r}$,

where q and r are the quotient and remainder respectively, when n is divisible by k.

(iv) Number of distinct or dissimilar terms in the expansion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \ldots + r_k = n$, i.e. ${}^{n+k-1}C_{k-1}$

Note If n is any positive integer, then number of terms in the expansion of $(x + y + z)^n$ is $\frac{(n+1)(n+2)}{2}$.

Similarly, number of terms in the expansion of $(w + x + y + z)^n$ is $\frac{(n+1)(n+2)(n+3)}{6}$.

@CLASS XII SYLLABUS

JEE Final Touch

Fast Track Revision

Application of Integrals

In geometry, we have learnt formulae to calculate areas of various geometrical figures such as triangles, rectangles, trapeziums and circles etc. By using geometry, we cannot find the area enclosed by many curves. The concept of definite integral can be used to find the areas enclosed by curves. Here, we will study a specific application of integral to find the area under the simple curves, area between lines and arcs of circles, parabolas and ellipse. Before starting this topic we overview some important concepts used in this topic such as curve tracing and transformation.

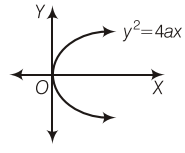
Curve Tracing

For the evaluation of area bounded by the curves, we should have very good understanding of tracing of curve. Following are important steps to be followed to trace the curve

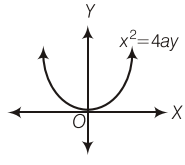
(i) Find the set of permissible values of x (i.e. domain).

(ii) Check the symmetry of the curve. This symmetry can be judged as follows

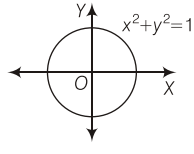
(a) If all the powers of y are even, then the curve is symmetrical about X-axis.



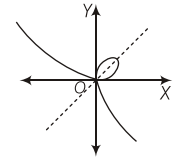
(b) If all the powers of x are even, then the curve is symmetrical about Y-axis.



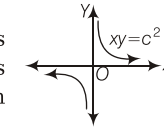
(c) If x and y both have even power, then curve is symmetrical along X-axis as well as Y-axis.



(d) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about y = x.



(e) If on interchanging the sign of x and y, there is no change in equation or equation remains unaltered, then the equation is symmetric in opposite quadrants.



(iii) Find the points, where the curve crosses the X-axis and also the Y-axis.

(iv) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve, where you have horizontal tangents.

(v) Find the period of the curve, if it is periodic.

(vi) Find the asymptote(s) of the curve, if any.

Asymptote is a line, whose distance from curve tends to zero as point on the curve moves towards infinity.

Read the following cases of asymptote

(a) If $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$, then $x = a$ is asymptote of $y = f(x)$.

(b) If $\lim_{x \to \infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$, then $y = k$ is asymptote of $y = f(x)$.

(c) If $\lim_{x \to \infty} \frac{f(x)}{x} = m_1$, $\lim_{x \to \infty} [f(x) - m_1 x] = c_1$, then $y = m_1 x + c_1$ is an asymptote inclined to right.

(d) If $\lim_{x \to -\infty} \frac{f(x)}{x} = m_2$, $\lim_{x \to -\infty} [f(x) - m_2 x] = c_2$, then $y = m_2 x + c_2$ is an asymptote inclined to left.

Important Transformations

	Original functions	Transformed functions	Process to transform original functions
(i)	$y = f(x)$	$y = -f(x)$	Flip about the X-axis
(ii)	$y = f(x)$	$y = f(-x)$	Flip about the Y-axis
(iii)	$y = f(x)$	$y = f(x) + k$	Shift the graph k units upwards or downwards depending on whether k is positive or negative.
(iv)	$y = f(x)$	$y = f(x + k)$	Shift left or shift right the graph by k units depending on whether k is positive or negative.
(v)	$y = f(x)$	$y = kf(x)$	Stretch or compress the graph along Y-axis k units depending on $\lvert k\rvert > 1$ or $\lvert k\rvert < 1$. If k is negative, also flip the curve about the X-axis.
(vi)	$y = f(x)$	$y = f(kx)$	Stretch or compress the graph along X-axis by k units, depending on $\lvert k\rvert < 1$ or $\lvert k\rvert > 1$, if k is negative, also flip it along the Y-axis.
(vii)	$y = f(x)$	$y = \lvert f(x)\rvert$	Portion of the graph $y = f(x)$ uneffected, if it is positive, take the mirror image of the negative part with respect to X-axis.
(viii)	$y = f(x)$	$y = f(\lvert x\rvert)$	Discard the left side of the graph (which lies left of Y-axis) and take mirror image of right side of the graph with respect to Y-axis.
(ix)	$y = f(x)$	$\lvert y\rvert = f(x)$	Discard the graph, which is below the X-axis and take the mirror image of the remaining graph with respect to X-axis.

AREA UNDER SIMPLE CURVES AND LINES

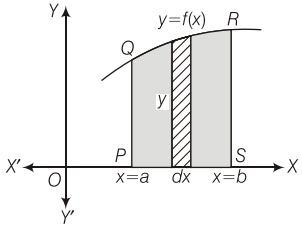
Area of Bounded Regions

Area under the curve can be assumed as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx. Then, area of elementary strip $(= dA) = y\,dx$, where $y = f(x)$ is a continuous function.

This area is called elementary area, which is located at an arbitrary position within the region, which is specified by some value of x between a and b.

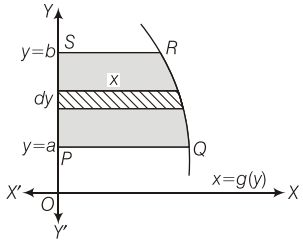
Some Different Cases of Bounded Area

(i) The area bounded by continuous curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$, where $b > a$, is given by



$$\text{Area} = \int_a^b dA = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

(ii) Area bounded by continuous curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$, where $b > a$, is given as Area $= \int_a^b dA = \int_a^b x\,dy = \int_a^b f(y)\,dy$



Sign Convention to Find Area

Following are the important cases to find the sign convention

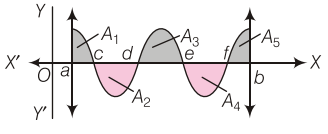
Case I In the expression $\int_a^b f(x)\,dx$, if $b > a$ and $f(x) > 0$, $\forall\, x \in [a, b]$, then this expression will give area enclosed by the curve $f(x)$, X-axis and the lines $x = a$ and $x = b$, which is positive. Here, you need not to modify anything.

Case II In the expression $\int_a^b f(x)\,dx$, if $b > a$ and

$$f(x) < 0,\ \forall\, x \in [a, b],\ \text{then} \int_a^b f(x)\,dx < 0$$

i.e. integrated value is negative, hence area is negative. To make it positive, we take its absolute value.

Case III In the expression $\int_a^b f(x)\,dx$, if $b > a$, $f(x)$ changes its sign number of time in the interval $[a, b]$. Then, we need to separate intervals in which $f(x) > 0$ and $f(x) < 0$. Observe the following figure which gives clear thought of it.



Required area,

$$\int_a^b f(x)\,dx = A_1 + |A_2| + A_3 + |A_4| + A_5$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \left|\int_c^d f(x)\,dx\right| + \int_d^e f(x)\,dx + \left|\int_e^f f(x)\,dx\right| + \int_f^b f(x)\,dx$$

SPEED BOOSTER

- The area bounded by $y^2 = 4ax$ and $y = mx$ is $\dfrac{8a^2}{3m^3}$ sq units.
- The area bounded by $y^2 = 4ax$ and its latusrectum is $8a^2/3$ sq units.
- The area bounded by one arc of $\sin ax$ or $\cos ax$ and X-axis is $2/a$ sq units.

Best Practice SHOTS

1. The area bounded by $y = xe^{|x|}$ and lines $|x| = 1, y = 0$ is

(a) 4 sq units (b) 6 sq units
(c) 1 sq unit (d) 2 sq units

2. The area bounded by the curve $y = x$, X-axis and ordinates $x = -1$ to $x = 2$ is [RPET 2001, AMU 2010]

(a) 0 sq unit (b) $\frac{1}{2}$ sq unit (c) $\frac{3}{2}$ sq units (d) $\frac{5}{2}$ sq units

3. The smaller area enclosed by the circle $x^2 + y^2 = 9$ and the line $x = 1$, is

(a) $(9\sec^{-1} 3 - \sqrt{8})$ sq unit (b) $(3\sec^{-1} 9 - \sqrt{8})$ sq unit
(c) $(3\sec^{-1} \sqrt{8} - 9)$ sq unit (d) $(9\sec^{-1} \sqrt{8} - 3)$ sq unit

4. The area (in sq units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X-axis and lying in the first quadrant, is [JEE Main 2013]

(a) 9 (b) 6 (c) 18 (d) $\frac{27}{4}$

5. Area enclosed by the curve $y = (x-1)(x-2)(x-3)$ lying between the ordinates $x = 0$ and $x = 3$, is

(a) $\frac{10}{4}$ sq units (b) $\frac{11}{4}$ sq units
(c) $\frac{4}{10}$ sq unit (d) None of these

6. The area between the curve $y = 2x^4 - x^2$, the X-axis and the ordinates of the two minima of the curve is

(a) $\frac{11}{60}$ sq unit (b) $\frac{7}{120}$ sq unit (c) $\frac{1}{30}$ sq unit (d) $\frac{7}{90}$ sq unit

7. The area of the closed figure bounded by $x = -1$, $x = 2$ and $y = \begin{cases} -x^2 + 2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$ and the abscissa axis is

(a) $\frac{16}{3}$ sq units (b) $\frac{10}{3}$ sq units (c) $\frac{13}{3}$ sq units (d) $\frac{7}{3}$ sq units

8. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals 9/2?

(a) -4 (b) -3 (c) 2 (d) 4

9. The area between the parabola $y = x^2$ and the line $y = x$ is [UPSEAT 2004, MPPET 2006, KCET 2004]

(a) $\frac{1}{6}$ sq unit (b) $\frac{1}{3}$ sq unit (c) $\frac{1}{2}$ sq unit (d) None

10. Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is [KCET 1993]

(a) πab sq units (b) $\frac{1}{2}\pi ab$ sq units
(c) $\frac{1}{4}\pi ab$ sq units (d) None of these

11. The area of the region whose boundaries are defined by the curves $y = 2\cos x$, $y = 3\tan x$ and the Y-axis is

(a) $\left[1 + 3\ln\left(\frac{2}{\sqrt{3}}\right)\right]$ sq units (b) $\left[1 + \frac{3}{2}\ln 3 - 3\ln 2\right]$ sq units
(c) $\left[1 + \frac{3}{2}\ln 3 - \ln 2\right]$ sq units (d) $[\ln 3 - \ln 2]$ sq units

12. If the areas S_0, S_1, S_2, ... bounded by the X-axis and half-waves of the curve $y = e^{-x}\sin x$ (where, $x \ge 0$), then value of S_0 is

(a) $\frac{1}{2}(1 + e^{\pi})$ sq unit (b) $\frac{1}{2}(1 + e^{-\pi})$ sq unit
(c) $\frac{1}{2}(1 - e^{-\pi})$ sq unit (d) $\frac{1}{2}(e^{\pi} - 1)$ sq unit

13. The area enclosed between the curve $y = \log_e(x + e)$ and the coordinate axes, is [AIEEE 2005]

(a) 4 sq units (b) 3 sq units (c) 2 sq units (d) 1 sq unit

14. The area of the region bounded by the parabola $(y-2)^2 = (x-1)$, the tangent to the parabola at the point (2, 3) and the X-axis is

(a) 3 sq units (b) 6 sq units (c) 9 sq units (d) 12 sq units

15. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, which of the following is incorrect? [IIT JEE 2012]

(a) $S \ge \frac{1}{e}$ (b) $S \ge 1 - \frac{1}{e}$
(c) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

16. The area of the curve $xy^2 = a^2(a - x)$ bounded by Y-axis is

(a) πa^2 sq units (b) $2\pi a^2$ sq units
(c) $3\pi a^2$ sq units (d) $4\pi a^2$ sq units

17. Let $f(x) = x^{2/3}$, $x \ge 0$. Then, the area of the region enclosed by the curve $y = f(x)$ and the three lines $y = x$, $x = 1$ and $x = 8$, is [WB JEE 2013]

(a) $\frac{63}{2}$ sq units (b) $\frac{93}{5}$ sq units (c) $\frac{105}{7}$ sq units (d) $\frac{129}{10}$ sq units

18. The volume of the solid generated by the revolution of the curve $y = \frac{a^3}{a^2 + x^2}$ about X-axis, is [UPSEAT 2005]

(a) $\frac{1}{2}\pi^3 a^2$ cu units (b) $\frac{1}{2}\pi^2 a^3$ cu units
(c) $\pi^3 a^2$ cu units (d) $\pi^2 a^3$ cu units

19. The area enclosed between the curve $y = 1 + x^2$, the Y-axis and the straight line $y = 5$ is given by [WB JEE 2008]

(a) $\frac{14}{3}$ sq units (b) $\frac{7}{3}$ sq units (c) 5 sq units (d) $\frac{16}{3}$ sq units

20. The measurement of the area bounded by the coordinate axes and the curve $y = \log_e x$ is

(a) 1 (b) 2 (c) 3 (d) ∞

21. The value of the parameter a such that the area bounded by $y = a^2x^2 + ax + 1$, coordinate axes and the line $x = 1$ attains its least value, is equal to

(a) $-\frac{1}{4}$ sq unit (b) $-\frac{1}{2}$ sq unit (c) $-\frac{3}{4}$ sq unit (d) -1 sq unit

22. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} x f(x)\, dx$ and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and the X-axis. Then, [IIT JEE 2011]

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$
(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

23. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive X-axis is [AIEEE 2011]

(a) $\frac{1}{2}$ sq unit (b) 1 sq unit (c) $\frac{3}{2}$ sq units (d) $\frac{5}{2}$ sq units

24. Let the straight line $x = b$ divides the area enclosed by $y = (1-x)^2$, $y = 0$ and $x = 0$ into two parts R_1 $(0 \le x \le b)$ and R_2 $(b \le x \le 1)$ such that $R_1 - R_2 = 1/4$, then b equals [IIT JEE 2011]

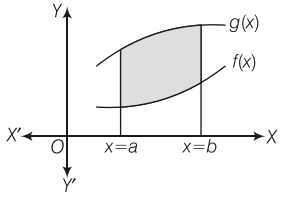
(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

AREA BETWEEN TWO CURVES

Area between $y = f(x)$ and $y = g(x)$

For finding area between given curves, we have following conditions

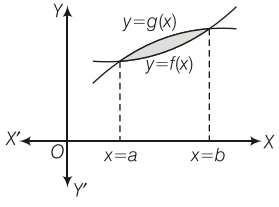
1. The area bounded by the straight lines $x = a$, $x = b$ $(a < b)$ and the curves $y = f(x)$ and $y = g(x)$ provided $f(x) \le g(x)$ (where, $a \le x \le b$) is given as



$$\text{Area} = \int_a^b [g(x) - f(x)]\, dx$$

2. When two curves $y = f(x)$ and $y = g(x)$ intersect, then bounded area is given by

$$\text{Area, } A = \int_a^b [g(x) - f(x)]\, dx$$

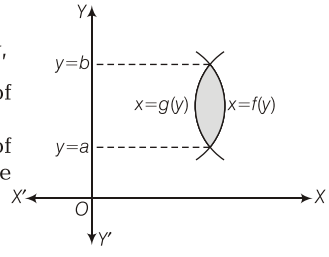


where, a and b are roots of the equation $f(x) = g(x)$ or x-coordinates of intersection points of the curves.

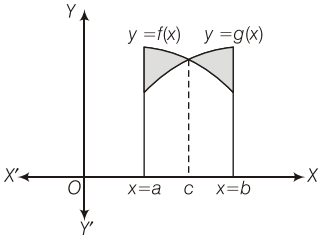
3. When two curves $x = f(y)$ and $x = g(y)$ intersect, then bounded area is given by

$$\text{Area} = A = \int_a^b [f(y) - g(y)]\, dy,$$

where, a and b are roots of the equation $f(y) = g(y)$ or y-coordinates of intersection points of the curves.



4. The area bounded by $y = f(x)$ and $y = g(x)$, (where $a \le x \le b$) when they intersect at $x = c \in (a, b)$ is given as



$$A = \int_a^b |f(x) - g(x)|\, dx$$

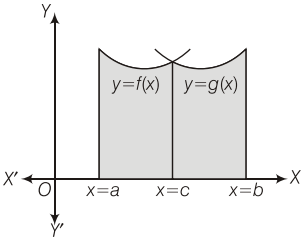
$$= \int_a^c [f(x) - g(x)]\, dx + \int_c^b [g(x) - f(x)]\, dx$$

Area Bounded by Two Curves and X-Axis

Area bounded by the two curves $y = f(x)$, $y = g(x)$ and X-axis between the lines $x = a$ and $x = b$ is given by

$$\text{Area} = A = \int_a^c f(x)\, dx + \int_c^b g(x)\, dx,$$

where c is the intersection point between $x = a$ and $x = b$.



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- The area of the region bounded by $y^2 = 4ax$ and $x^2 = 4by$, where $a > 0$ and $b > 0$, is $16ab / 3$ sq units.
- Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq units.

Best Practice SHOTS

25. The area of the region
$R = \{(x, y) : |x| \le |y| \text{ and } x^2 + y^2 \le 1\}$, is
(a) $\frac{3\pi}{8}$ sq units *(b)* $\frac{5\pi}{8}$ sq units *(c)* $\frac{\pi}{2}$ sq units *(d)* $\frac{\pi}{8}$ sq unit

26. The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is
(a) 4π sq units *(b)* 8π sq units
(c) 4 sq units *(d)* 8 sq units

27. The area bounded by $y = \sec^{-1} x$, $y = \text{cosec}^{-1} x$ and line $x - 1 = 0$, is
(a) $\left[\log(3 + 2\sqrt{2}) - \frac{\pi}{2}\right]$ sq unit
(b) $\left[\frac{\pi}{2} - \log(3 + 2\sqrt{2})\right]$ sq unit
(c) $\pi - \log 3$ sq unit
(d) None of the above

28. The area of the region in Ist quadrant bounded by the Y-axis, $y = \frac{x}{4}$, $y = 1 + \sqrt{x}$ and $y = \frac{2}{\sqrt{x}}$ is
(a) $\frac{2}{3}$ sq unit *(b)* $\frac{11}{3}$ sq units *(c)* $\frac{8}{3}$ sq units *(d)* $\frac{13}{6}$ sq units

29. Area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$, is
(a) $\left|\frac{3\pi - 8}{3}\right|$ sq unit *(b)* $\left|\frac{\pi - 8}{3}\right|$ sq unit
(c) $\left|\frac{2\pi - 8}{3}\right|$ sq unit *(d)* None of these

30. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If S_1, S_2 and S_3 are respectively the areas of these parts numbered from top to bottom, then $S_1 : S_2 : S_3$ is **[AIEEE 2006]**
(a) 1 : 1 : 1 *(b)* 2 : 1 : 2 *(c)* 1 : 2 : 3 *(d)* 1 : 3 : 2

31. The area of the region containing the points (x, y) satisfying $4 \le x^2 + y^2 \le 2(|x| + |y|)$ is
(a) 4π sq units *(b)* 2 sq units
(c) 2π sq units *(d)* 8 sq units

32. The area enclosed by the curves $xy^2 = a^2(a - x)$ and $(a - x)y^2 = a^2 x$ is
(a) $\pi a^2 / 3$ sq unit *(b)* $(4 - \pi)a^2$ sq unit
(c) $(\pi - 2)a^2$ sq unit *(d)* None of these

33. The area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is
(a) $4\sqrt{3}$ sq units *(b)* $6\sqrt{3}$ sq units
(c) $8\sqrt{3}$ sq units *(d)* $12\sqrt{3}$ sq units

34. The area bounded by the curve $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is **[AIEEE 2010]**
(a) $(4\sqrt{2} - 1)$ sq units *(b)* $(4\sqrt{2} + 1)$ sq units
(c) $(4\sqrt{2} - 2)$ sq units *(d)* $(4\sqrt{2} + 2)$ sq units

35. The area bounded by curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y = \frac{1}{4}$ is **[IIT JEE 2005]**
(a) $\frac{1}{3}$ sq unit *(b)* $\frac{2}{3}$ sq unit *(c)* $\frac{1}{4}$ sq unit *(d)* $\frac{1}{5}$ sq unit

36. The ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$, is
(a) $\frac{147}{45}$ *(b)* $\frac{48}{146}$ *(c)* $\frac{146}{48}$ *(d)* $\frac{45}{147}$

37. The area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$ is
(a) 1 sq unit *(b)* $\frac{4}{3}$ sq units *(c)* $\frac{2}{3}$ sq unit *(d)* 2 sq units

38. Area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is
(a) $(e - 2)$ sq unit *(b)* $(3 - e)$ sq unit
(c) e sq unit *(d)* $(e - 1)$ sq unit

39. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is **[JEE Advanced 2014]**
(a) $4(\sqrt{2} - 1)$ sq unit *(b)* $2\sqrt{2}(\sqrt{2} - 1)$ sq unit
(c) $2(\sqrt{2} + 1)$ sq unit *(d)* $2\sqrt{2}(\sqrt{2} + 1)$ sq unit

40. If two curves $C_1 \equiv [f(y)]^{2/3} + [f(x)]^{1/3} = 0$ and $C_2 \equiv [f(y)]^{2/3} + [f(x)]^{2/3} = 12$, satisfying the relation $(x - y) f(x + y) - (x + y) f(x - y) = 4xy \quad (x^2 - y^2)$, then area bounded by C_1 and C_2 is
(a) $(2\pi - \sqrt{3})$ sq units *(b)* $(\pi + \sqrt{6})$ sq units
(c) $(2\pi + \sqrt{3})$ sq units *(d)* $(2\sqrt{3} - \pi)$ sq unit

41. The area in the first quadrant between $x^2 + y^2 = \pi^2$ and $y = \sin x$ is
(a) $\frac{(\pi^3 - 8)}{4}$ sq unit *(b)* $\frac{\pi^3}{4}$ sq unit
(c) $\frac{(\pi^3 - 16)}{4}$ sq unit *(d)* $\frac{(\pi^3 - 8)}{2}$ sq unit

42. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is **[JEE Main 2014]**
(a) $\left(\frac{\pi}{2} + \frac{4}{3}\right)$ sq unit *(b)* $\left(\frac{\pi}{2} - \frac{4}{3}\right)$ sq unit
(c) $\left(\frac{\pi}{2} - \frac{2}{3}\right)$ sq unit *(d)* $\left(\frac{\pi}{2} + \frac{2}{3}\right)$ sq unit

43. Area of the region bounded by the curves $y = 2^x$, $y = 2x - x^2$ and straight lines $x = 0$ and $x = 2$ is given by

(a) $\left(\frac{3}{\log 2} - \frac{4}{3}\right)$ sq unit (b) $\left(\frac{3}{\log 2} + \frac{4}{3}\right)$ sq unit

(c) $\left(3 \log 2 - \frac{4}{3}\right)$ sq unit (d) None of these

44. The area induced between the curves $y = \frac{x^2}{4a}$ and $y = \frac{8a^3}{x^2 + 4a^2}$ is given by

(a) $a^2\left(2\pi - \frac{4}{3}\right)$ sq unit (b) $a^2\left(\pi - \frac{4}{3}\right)$ sq unit

(c) $a^2\left(2\pi + \frac{1}{3}\right)$ sq unit (d) $a^2\left(\pi + \frac{4}{3}\right)$ sq unit

45. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then, the area of the quadrilateral $PQRS$ is **[JEE Advanced 2014]**

(a) 3 sq units (b) 6 sq units (c) 9 sq units (d) 15 sq units

46. The area of the region bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$, is

(a) $\frac{1}{3}$ sq unit (b) $\frac{2}{3}$ sq unit (c) $\frac{4}{3}$ sq units (d) $\frac{5}{3}$ sq units

47. The area bounded by the curves $y = \sin^{-1}|\sin x|$ and $y = (\sin^{-1}|\sin x|)^2$, where $0 \le x \le 2\pi$, is

(a) $\left(\frac{1}{3} + \frac{\pi^2}{4}\right)$ sq unit (b) $\left(\frac{1}{6} + \frac{\pi^3}{8}\right)$ sq unit

(c) 2 sq units (d) None of these

48. If area of the region bounded by the curves $y = 1 + \cos x$, $y = 1 + \cos(x - \alpha)$ and $x = 0$ is same as that of the region bounded by $y = 1 + \cos(x - \alpha)$, $y = 1$ and $x = \pi$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $x \in [0, \pi]$, then the value of α is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

49. Consider two curves $C_1 : y^2 = 4[\sqrt{y}]x$ and $C_2 : x^2 = 4[\sqrt{x}]y$, where $[\cdot]$ denotes the greatest integer function. Then, the area of region enclosed by these two curves within the square formed by the lines $x = 1$, $y = 1$, $x = 4$, $y = 4$ is

(a) $\frac{11}{4}$ sq units (b) $\frac{11}{3}$ sq units (c) $\frac{10}{3}$ sq units (d) $\frac{8}{3}$ sq units

Answers

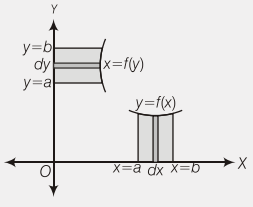
1. (d)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (b)	**6.** (b)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (a)
11. (b)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (c)	**16.** (a)	**17.** (d)	**18.** (b)	**19.** (a)	**20.** (d)
21. (c)	**22.** (c)	**23.** (c)	**24.** (b)	**25.** (c)	**26.** (d)	**27.** (a)	**28.** (b)	**29.** (a)	**30.** (a)
31. (d)	**32.** (c)	**33.** (d)	**34.** (c)	**35.** (a)	**36.** (d)	**37.** (d)	**38.** (b)	**39.** (b)	**40.** (c)
41. (a)	**42.** (a)	**43.** (a)	**44.** (a)	**45.** (d)	**46.** (c)	**47.** (d)	**48.** (c)	**49.** (b)	

Formulae at a Glance

1.

(i) Area between the curve $y = f(x)$, X-axis and two ordinates at the points $x = a$ and $x = b$ is $A = \int_a^b f(x)dx$.

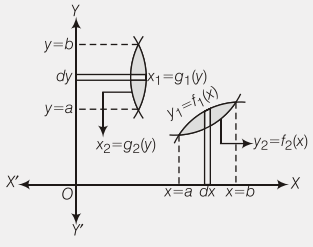
(ii) Area between the curve $x = f(y)$, Y-axis and $y = a$, $y = b$ is $A = \int_a^b f(y)dy$.



2.

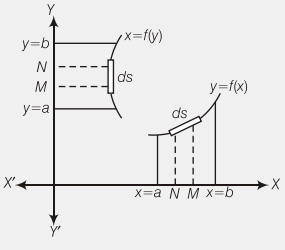
(i) If the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ intersect at two points $x = a$ and $x = b$, then common area $A = \int_a^b [f_1(x) - f_2(x)]dx$.

(ii) If the two curves $x_1 = g_1(y)$ and $x_2 = g_2(y)$ intersect at two points $y = a$ and $y = b$, then common area $A = \int_a^b [g_1(y) - g_2(y)]dy$.



3. Area bounded by the curve $r = f(\theta)$ and radii $\theta = \alpha$, $\theta = \beta$ $(\alpha < \beta)$ is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$.

4. The curved surface area of the solid generated by the revolution about the X-axis (or Y-axis), of the area bounded by the curve $y = f(x)$ (or $x = f(y)$), $x = a$, $x = b$ (or $y = a$, $y = b$) and the X-axis (or Y-axis) is $2\pi\int_{x=a}^b f(x)\,ds$



or $2\pi\int_{y=a}^b f(y)\,ds$, where

ds (Arc of length) $= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$ or $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy$.

5. Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

6. Area of region bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16ab}{3}$ sq units.

7. Area of region bounded by $y^2 = 4ax$ and $y = mx$ is $\frac{8a^2}{3m^3}$.

Differential Equation

A differential equation is a mathematical equation that relates some function with its derivative. It is one of the important topic which is frequently asked in JEE Main and Advanced. Atleast one question is asked every year from this topic. For convenience, we are dividing this topic into four sub-topics *viz*., 'Order, degree and formation of differential equation', 'Solution of differential equation', 'First order linear differential equation and application of first order differential equation'. Let us discuss these topics.

ORDER, DEGREE AND FORMATION OF DIFFERENTIAL EQUATION

Differential Equation

An equation involving independent variable(s), dependent variable and derivative(s) of dependent variable with respect to independent variable(s) is called a differential equation.

e.g. $$2x\frac{dy}{dx} - 3y = 5$$

and $$4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

are differential equations.

Note $x^2 + 2xy + y^2 = 0$ is not a differential equation.

Types of Differential Equation

There are two types of differential equation, which are given as

(i) **Ordinary differential equation** A differential equation involving derivatives with respect to a single independent variable, is called an ordinary differential equation.

(ii) **Partial differential equation** A differential equation involving derivatives with respect to more than one independent variables, is called a partial differential equation.

But here we will only discuss ordinary differential equation.

Order of Differential Equation

In a differential equation, the order of highest derivative occurring in it is called the order of differential equation.

Degree of Differential Equation

When an equation is a polynomial in derivative(s), the power of highest order derivative is known as degree of differential equation.

e.g. To find degree of differential equation

$$\sqrt{\frac{d^3y}{dx^3}} = \sqrt[4]{\left(\frac{dy}{dx} + 5\right)},$$

we convert it in polynomial form, by removing its radicals, i.e. write given equation as

$$\left(\sqrt{\frac{d^3y}{dx^3}}\right)^4 = \left(\sqrt[4]{\frac{dy}{dx} + 5}\right)^4$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{dy}{dx} + 5$$

Hence, given differential equation has order 3 (order of highest derivative) and degree 2 (power of highest order derivative).

Formation of Differential Equation

Consider a equation $f(x, y, c_1, c_2, ..., c_n)$ containing n independent arbitrary constants $c_1, c_2, ..., c_n$ besides the variables x and y.

The formation of a differential equation representing given equation, means to form a differential equation by eliminating arbitrary constant from the given equation. For this, we differentiate $f(x, y, c_1, c_2, ..., c_n)$ n times, we shall get n equations.

Now, among these n equations and the given equation, in all $(n+1)$ equations, if the n arbitrary constants $c_1, c_2, c_3, ..., c_n$ are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

Working Rule

***Step* I** Obtained the number of arbitrary constant present in given equation.

***Step* II** Differentiate the given equation n times with respect to x, where n is number of arbitrary constant present.

***Step* III** Eliminate arbitrary constant using $(n + 1)$ equations.

Best Practice SHOTS

1. The degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^{2/3} + 4 - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$ is

(a) 1 (b) 2
(c) 3 (d) None of these

2. If p and q are the order and degree of the differential equation $y\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} + xy = \cos x$, then

(a) $p < q$ (b) $p = q$
(c) $p > q$ (d) None of these

3. The degree and order of differential equation $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$, are respectively

(a) 4, 3 (b) 3, 4
(c) undefined, 4 (d) None of these

4. The order and degree of differential equation $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$ are respectively

(a) 1, 2 (b) 1, undefined
(c) 1, 1 (d) None of these

5. If m and n are the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\frac{d^3y}{dx^3}} + \frac{d^3y}{dx^3} = x^2 - 1$, then

(a) $m = 3, n = 3$ (b) $m = 3, n = 2$
(c) $m = 3, n = 5$ (d) $m = 3, n = 1$

6. Which of the following differential equations has the same order and degree? **[Kurukshetra CEE 1998]**

(a) $\frac{d^4y}{dx^4} + 8\left(\frac{dy}{dx}\right)^6 + 5y = e^x$

(b) $5\left(\frac{d^3y}{dx^3}\right)^4 + 8\left(1 + \frac{dy}{dx}\right)^2 + 5y = x^8$

(c) $\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{2/3} = 4\frac{d^3y}{dx^3}$

(d) $y = x^2\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

7. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}$ determines, a family of circles with **[IIT JEE 2007]**

(a) variable radii and a fixed centre at (0, 1)
(b) variable radii and a fixed centre at (0, – 1)
(c) fixed radius 1 and variable centres along the X-axis
(d) fixed radius 1 and variable centres along the Y-axis

8. The differential equation that represents all parabolas having their axis of symmetry coincident with the X-axis is

(a) $yy_1^2 + y_2 = 0$ (b) $yy_2 + y_1^2 = 0$
(c) $y_1^2 - yy_2 = 0$ (d) $yy_2 + y_1 = 0$

9. The differential equation of all circles passing through the origin and having their centres on the X-axis, is **[AIEEE 2007]**

(a) $y^2 = x^2 + 2xy\frac{dy}{dx}$ (b) $y^2 = x^2 - 2xy\frac{dy}{dx}$
(c) $x^2 = y^2 + xy\frac{dy}{dx}$ (d) $x^2 = y^2 + 3xy\frac{dy}{dx}$

10. The differential equation of the family of lines through the origin, is **[Delhi CEE 2002]**

(a) $x\frac{dy}{dx} - y = 0$ (b) $x + \frac{dy}{dx} = 0$
(c) $\frac{dy}{dx} = y$ (d) $\frac{dy}{dx} = x$

11. The differential equation of the family of curves represented by the equation $x^2 + y^2 = a^2$ is **[WB JEE 2011]**

(a) $x + y\frac{dy}{dx} = 0$ (b) $y\frac{dy}{dx} = x$
(c) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ (d) None of these

12. The differential equation of displacement of all "Simple harmonic motion" of given period $2\pi/n$, is

(a) $\frac{d^2x}{dt^2} + nx = 0$ (b) $\frac{d^2x}{dt^2} + n^2x = 0$
(c) $\frac{d^2x}{dt^2} - n^2x = 0$ (d) $\frac{d^2x}{dt^2} + \frac{1}{n^2}x = 0$

13. The differential equation of all non-horizontal lines in a plane is

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2y}{dx^2} =$ constant
(c) $\frac{d^2x}{dy^2} = 0$ (d) None of these

14. The differential equation of all non-vertical lines in a plane is

(a) $\frac{d^2y}{dx^2} =$ constant (b) $\frac{d^2y}{dx^2} = 0$
(c) $\frac{d^2x}{dy^2} = 0$ (d) $\frac{d^2x}{dy^2} =$ constant

15. Which of the following is not the differential equation of family of curves whose tangent form an angle of $\pi/4$ with the hyperbola $xy = c^2$?

(a) $\frac{dy}{dx} = \frac{x - y}{x + y}$ (b) $\frac{dy}{dx} = \frac{x}{x - y}$
(c) $\frac{dy}{dx} = \frac{x + y}{y - x}$ (d) None of these

SOLUTION OF DIFFERENTIAL EQUATION

Solution of Differential Equation

Any relation between dependent and independent variable, which when substituted in the differential equation reduce it to in an identity, is called a solution of differential equation. A solution of differential equation does not contain the derivative of the dependent variable with respect to independent variable.

Solutions of differential equation are of following two types

(i) **General solution** The solution of an ordinary differential equation of nth order, which contains n arbitrary constant, is called the general solution.

Let equation involving the variables x, y and n independent arbitrary constant be $f(x, y, c_1, c_2, \ldots, c_n) = 0$ and differential equation obtained from it, is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}\right) = 0, \text{ then}$$

$f(x, y, c_1, \ldots, c_n)$ is called general solution.

(ii) **Particular solution** Any solution obtained from the general solution of differential equation by giving particular values to the arbitrary constant, is called particular solution.

Various Methods to Solve First Order and First Degree Differential Equation

Here, we are giving various methods to solve first order and first degree differential equation.

Variable Separable Type

If a first order first degree equation can be expressed in such a manner that coefficient of dx is only a function of x, say $f(x)$, and coefficient of dy is only function of y, say $g(y)$, i.e. written in the form $f(x)\,dx + g(y)\,dy = 0$, to solve such differential equation we integrate both sides,

i.e. $\int f(x)\,dx + \int g(y)\,dy = C$

Reducible to Variable Separable Form

Differential equation of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by substituting $ax + by + c = z$.

$$\Rightarrow \quad a + b\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b}$$

Now, given equation becomes

$$\frac{\frac{dz}{dx} - a}{b} = f(z)$$

$$\Rightarrow \quad \frac{dz}{dx} = b\,f(z) + a$$

$$\Rightarrow \quad \frac{dz}{b\,f(z) + a} = dx$$

On integrating of above equation, we get

$$\int \frac{dz}{b\,f(z) + a} = x + C, \text{ where } z = ax + by + c$$

which represents the general solution of the differential equation.

SPEED BOOSTER

If we can write the differential equation in the form $f[f_1(x, y)d\{f_1(x, y)\}] + \phi[f_2(x, y)d\{f_2(x, y)\}] + \ldots = 0$, then each term can be easily integrated separately. For this following results must be memorised

- $d(x \pm y) = dx \pm dy$
- $d(xy) = xdy + ydx$
- $xdx + ydy = \frac{1}{2}d(x^2 + y^2)$
- $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$
- $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$
- $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$
- $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$
- $d\left\{\ln\left(\frac{x}{y}\right)\right\} = \frac{ydx - xdy}{xy}$
- $d\left\{\ln\left(\frac{y}{x}\right)\right\} = \frac{xdy - ydx}{xy}$
- $d\{\ln(xy)\} = \frac{xdy + ydx}{xy}$
- $d\{\ln(x + y)\} = \frac{dx + dy}{x + y}$
- $d\left\{\frac{1}{2}\ln(x^2 + y^2)\right\} = \frac{xdx + ydy}{x^2 + y^2}$
- $d\left(\frac{-1}{xy}\right) = \frac{xdy + ydx}{x^2y^2}$
- $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$
- $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$
- $d(x^my^n) = x^{m-1}y^{n-1}(mydx + nxdy)$
- $\frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{[f(x,y)]^n}$

Homogeneous Differential Equation

A function $f(x,y)$ is called homogeneous function of degree n, if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

e.g. $f(x, y) = x^3y - y^3x$ is homogeneous function of degree 4, since $f(\lambda x, \lambda y) = \lambda^3x^3\ \lambda y - \lambda^3y^3\ \lambda x$

$$f(\lambda x, \lambda y) = \lambda^4\ (x^3y - y^3x) = \lambda^4 f(x, y)$$

A differential equation of the form $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is a homogeneous function of degree zero, is called a homogeneous differential equation. Such equation is solved by substituting $y = vx$ or $x = yv$ according as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ or $\frac{dy}{dx} = G\left(\frac{x}{y}\right)$ and then separate the variables.

Reducible to Homogeneous Form

Equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f}$ $(ae \neq bd)$ can be reduced to a homogeneous form by changing the variables x, y to X, Y by the equations $x = X + h$, $y = Y + k$, where h, k are constants and are to be chosen such that the given equation becomes homogeneous. On substituting $x = X + h$ and $y = Y + k$, we get

$$\frac{dy}{dx} = \frac{d(Y + k)}{dx} = \frac{dY}{dx} = \frac{dY}{dX} \cdot \frac{dX}{dx} = \frac{dY}{dX}$$

So given equation becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{dX + eY + (dh + ek + f)}$$

Now, h and k be chosen such that $ah + bk + c = 0$ and $dh + ek + f = 0$. This gives $h = \frac{bf - ec}{ae - bd}$, $k = \frac{cd - af}{ae - bd}$

Thus, given equation reduced to $\frac{dY}{dX} = \frac{aX + bY}{dX + eY}$

which can be solved by putting $Y = vX$.

Best Practice SHOTS

16. If $f(x)$ and $g(x)$ are two solutions of the differential equation $a\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = e^x$, then $f(x) - g(x)$ is the solution of

(a) $a^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x$ (b) $a^2\frac{d^2y}{dx^2} + y = e^x$

(c) $a\frac{d^2y}{dx^2} + y = e^x$ (d) $a\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$

17. Solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is [Delhi CEE 1996]

(a) $y = 2$ (b) $y = 2x$

(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

18. The solution of the differential equation $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$, is

(a) $y = \tan^{-1} x$ (b) $y - x = C(1 + xy)$

(c) $x = \tan^{-1} y$ (d) $\tan(xy) = C$

19. Solution of the differential equation $xdy - ydx = 0$ represents

(a) a rectangular hyperbola

(b) a straight line passing through the origin

(c) parabola whose vertex is at the origin

(d) circle whose centre is at the origin

20. The straight line which satisfies the differential equation $\frac{dy}{dx} = m$ and cuts an intercept 3 on the positive side of Y-axis is [MPPET 2011]

(a) $y = mx + c$ (b) $y = mx + 3$

(c) $y = mx - 3$ (d) $y = -mx + 3$

21. The solution of differential equation $\frac{dy}{dx} = \frac{(1 + x)y}{(y - 1)x}$ is [Odisha JEE 2008]

(a) $\log xy + x + y = C$ (b) $\log\left(\frac{x}{y}\right) + x - y = C$

(c) $\log xy + x - y = C$ (d) None of these

22. The solution of the differential equation $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$ is [MPPET 1993]

(a) $\tan y = C(1 - e^x)^3$ (b) $(1 - e^x)^3 \tan y = C$

(c) $\tan y = C(1 - e^x)$ (d) $(1 - e^x)\tan y = C$

23. The general solution of the differential equation $\frac{dy}{dx} = e^y(e^x + e^{-x} + 2x)$ is [Kerala (Engineering) 2011]

(a) $e^{-y} = e^x - e^{-x} + x^2 + C$

(b) $e^{-y} = e^{-x} - e^x - x^2 + C$

(c) $e^{-y} = -e^{-x} - e^x - x^2 + C$

(d) $e^y = e^{-x} + e^x + x^2 + C$

24. The solution of $xy\frac{dy}{dx} = \frac{(1 + y^2)(1 + x + x^2)}{(1 + x^2)}$ is

(a) $\frac{1}{2}\log(1 + y^2) = \log x - \tan^{-1} x + C$

(b) $\frac{1}{2}\log(1 + y^2) = \log x + \tan^{-1} x + C$

(c) $\log(1 + y^2) = \log x - \tan^{-1} x + C$

(d) $\log(1 + y^2) = \log x + \tan^{-1} x + C$

25. Solution of the differential equation

$$x = 1 + xy\frac{dy}{dx} + \frac{(xy)^2}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{(xy)^3}{3!}\left(\frac{dy}{dx}\right)^3 + \ldots \text{ is}$$

(a) $y = \log_e x + C$ (b) $y = (\log_e x)^2 + C$

(c) $y = \pm\sqrt{(\log_e x)^2 + 2C}$ (d) $xy = x^y + C$

26. If $y = y(x)$ and $\frac{2 + \sin x}{y + 1}\left(\frac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ equals [IIT JEE 2004]

(a) 1/3 (b) 2/3 (c) $-1/3$ (d) 1

27. The solution of differential equation $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ is

(a) $\tan(x + y) + \sec(x + y) = x + C$

(b) $\tan(x + y) - \sec(x + y) = x + C$

(c) $\tan(x + y) + \sec(x + y) + x + C = 0$

(d) None of the above

28. The equation of a curve passing through the origin and satisfying the differential equation $\frac{dy}{dx} = (x - y)^2$, is

(a) $e^{2x}(1 - x + y) = 1 + x - y$
(b) $e^{2x}(1 + x - y) = 1 - x + y$
(c) $e^{2x}(1 - x + y) + (1 + x - y) = 0$
(d) $e^{2x}(1 + x + y) = 1 - x + y$

29. Solution of the differential equation $\left(\frac{x + y - 1}{x + y - 2}\right)\frac{dy}{dx} = \left(\frac{x + y + 1}{x + y + 2}\right)$, given that $y = 1$ when $x = 1$, is

(a) $\log\left|\frac{(x - y)^2 - 2}{2}\right| = 2(x + y)$

(b) $\log\left|\frac{(x - y)^2 + 2}{2}\right| = 2(x - y)$

(c) $\log\left|\frac{(x + y)^2 + 2}{2}\right| = 2(x - y)$

(d) None of the above

30. The general solution of the differential equation $xdy + y\,dx + 2x^3\,dx = 0$, is

(a) $xy + x^4 = C$ (b) $xy + \frac{1}{2}x^4 = C$

(c) $x + y + 1/2\,x^4 = 0$ (d) $xy - \left(\frac{1}{2}\right)x^4 = C$

31. Solution of the differential equation $(y + x\sqrt{xy}\,(x + y))\,dx + (y\sqrt{xy}\,(x + y) - x)\,dy = 0$ is

(a) $\frac{x^2 + y^2}{2} + \tan^{-1}\sqrt{\frac{y}{x}} = C$

(b) $\frac{x^2 + y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = C$

(c) $\frac{x^2 + y^2}{2} + 2\cot^{-1}\sqrt{\frac{x}{y}} = C$

(d) None of the above

32. The solution of differential equation $2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$, given $y(1) = \sqrt{\frac{\pi}{2}}$, is

(a) $\sin x^2y^2 = e^{x-1}$ (b) $\sin(x^2y^2) = x$
(c) $\cos(x^2y^2) + x = 0$ (d) $\sin(x^2y^2) = e^{x+1}$

33. The solution of the differential equation $(1 + xy)\,x\,dy + (1 - xy)y\,dx = 0$, is

(a) $\frac{1}{xy} + \log\left(\frac{y}{x}\right) = C$ (b) $-xy + \log\left(\frac{y}{x}\right) = C$

(c) $-\frac{1}{xy} + \log\left(\frac{y}{x}\right) = C$ (d) $-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = C$

34. Which of the following is a homogeneous differential equation?

(a) $(x - y)^2\frac{dy}{dx} = a^2$ (b) $x\frac{dy}{dx} - 2y = x^3$
(c) $(x + y - 1)\,dy - (x - y + 1)\,dx = 0$
(d) $x\sin\left(\frac{y}{x}\right)dy = \left\{y\sin\left(\frac{y}{x}\right) - x\right\}dx$

35. The differential equation $\frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}$ has the solution

(a) $x = y(\log x + C)$ (b) $y = x(\log y + C)$
(c) $x = (y + C)\log x$ (d) $y = (x + C)\log y$

36. The solution of differential equation $yy' = x\left\{\frac{y^2}{x^2} + \frac{f(y^2/x^2)}{f'(y^2/x^2)}\right\}$ is

(a) $f(y^2/x^2) = Cx^2$ (b) $x^2 f(y^2/x^2) = C^2y^2$
(c) $x^2f(y^2/x^2) = C$ (d) $f(y^2/x^2) = Cy/x$

37. The slope of the tangent at (x, y) to a curve passing through a point $(2, 1)$ is $\frac{x^2 + y^2}{2xy}$, then the equation of the curve is

(a) $2(x^2 - y^2) = 3x$ (b) $2(x^2 - y^2) = 6y$
(c) $x(x^2 - y^2) = 6$ (d) $x(x^2 + y^2) = 10$

FIRST ORDER LINEAR DIFFERENTIAL EQUATION

Differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone, is called linear differential equation.

Working Rule

Step I Write the given equation in the form $\frac{dy}{dx} + Py = Q$ and identify P and Q.

Step II Find integrating factor $IF = e^{\int P dx}$.

Step III The required solution is given by

$$y \cdot IF = \int Q \cdot IF\,dx + C$$

Note Sometimes a linear differential equation can be put in the form $\frac{dx}{dy} + Rx = S$, where R and S are functions of y alone, then its solution will be $x \cdot e^{\int R dy} = \int S \cdot e^{\int R dy} dy$

Bernoulli's Differential Equation

Sometimes a differential equation is not linear but it can be converted into a linear differential equation by a suitable substitution.

An equation of the form $\frac{dy}{dx} + Py = Qy^n$. $(n \neq 0, 1)$

where, P and Q are functions of x only, is known as Bernoulli's differential equation (for $n = 0$ the equation is linear and for $n = 1$ the equation is in variable separable form).

It is easy to reduce the above equation into linear form as below

Dividing both sides by y^n, we get $\frac{1}{y^n}\cdot\frac{dy}{dx}+\frac{1}{y^{n-1}}\cdot P = Q$

On putting $y^{1-n} = z$ and hence, $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$, the equation becomes $\frac{dz}{dx}+(1-n)Pz = (1-n)Q$ which is linear in z.

Here, IF $= e^{\int (1-n)\,P\,dx}$ and the solution is

$$z\cdot e^{\int (1-n)P\,dx} = \int\{(1-n)\cdot Q\cdot e^{(1-n)Pdx}\}dx + C$$

REMEMBER

- Another type of equation which is reducible to the linear form is $f'(y)\frac{dy}{dx} + P(x)\cdot f(y) = Q(x)$.
- An equation of this type can be easily reduced to linear form by taking $z = f(y)$.

Best Practice SHOTS

38. The integrating factor of the differential equation $(1-x^2)\frac{dy}{dx} - xy = 1$, is

(a) $-x$ (b) $\frac{x}{1+x^2}$

(c) $\sqrt{1-x^2}$ (d) $\frac{1}{2}\log(1-x^2)$

39. If integrating factor of $x(1-x^2)\,dy + (2x^2\,y - y - ax^3)\,dx = 0$ is $e^{\int P dx}$, then P is equal to

(a) $\frac{2x^2-ax^3}{x(1-x^2)}$ (b) $2x^3-1$

(c) $\frac{2x^2-1}{ax^3}$ (d) $\frac{2x^2-1}{x(1-x^2)}$

40. Which of the following transformations reduce the differential equation $\frac{dz}{dx}+\frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ into the form $\frac{du}{dx} + P(x)\,u = Q(x)$?

(a) $u = \log x$ (b) $u = e^z$

(c) $u = (\log z)^{-1}$ (d) $u = (\log z)^2$

41. Integrating factor of equation $(x^2+1)\frac{dy}{dx} + 2xy = x^2 - 1$ is [UPSEAT 2002; MPPET 2011]

(a) x^2+1 (b) $\frac{2x}{x^2+1}$

(c) $\frac{x^2-1}{x^2+1}$ (d) None of these

42. Solution of the equation $\cos^2 x\frac{dy}{dx} - (\tan 2x)\,y = \cos^4 x$, $|x| < \frac{\pi}{4}$, when $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$, is

(a) $y = \tan 2x\cos^2 x$

(b) $y = \cot 2x\cos^2 x$

(c) $y = \frac{1}{2}\tan 2x\cos^2 x$

(d) $y = \frac{1}{2}\cot 2x\cos^2 x$

43. If $\phi(x)$ is a differentiable function, then the solution of the differential equation

$dy + \{y\,\phi'(x) - \phi(x)\,\phi'(x)\}\,dx = 0$, is

(a) $y = \{\phi(x) - 1\} + Ce^{-\phi(x)}$

(b) $y\,\phi(x) = \{\phi(x)\}^2 + C$

(c) $y\,e^{\phi(x)} = \phi(x)\,e^{\phi(x)} + C$

(d) $y - \phi(x) = \phi(x)\,e^{-\phi(x)} + C$

44. The solution of the equation $(x+2y^3)\frac{dy}{dx} - y = 0$ is [MPPET 1998, 2002, 11, 12; Odisha JEE 2012]

(a) $y(1-xy) = Ax$

(b) $y^3 - x = Ay$

(c) $x(1-xy) = Ay$

(d) $x(1+xy) = Ay$

where, A is any arbitrary constant

45. The equation of the X-axis curve which is such that the portion of X-axis cut-off between the origin and tangent at any point is proportional to the ordinate of that point is

(a) $x = y(a - b\log y)$

(b) $\log x = by^2 + a$

(c) $x^2 = y(a - b\log y)$

(d) None of the above

(where, b is a constant of proportionality)

46. The differential equation $\frac{dy}{dx} = \frac{1}{xy\,[x^2\sin y^2 + 1]}$ has the solution

(a) $x^2[\cos y^2 - \sin y^2 - 2Ce^{-y^2}] = 2$

(b) $y^2[\cos x^2 - \sin y^2 - 2Ce^{-y^2}] = 2$

(c) $x^2[\cos y^2 - \sin y^2 - e^{-y^2}] = 4C$

(d) None of the above

47. The solution of the differential equation $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$ satisfying $y(1) = 1$, is

(a) $y^2 = x(1+x) - 10$ (b) $(x-2)^2 + (y-3)^2 = 5$

(c) a hyperbola (d) a circle

APPLICATION OF FIRST ORDER DIFFERENTIAL EQUATION

Following are important applications of differential equation

(i) **Growth and decay problem** Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that $\frac{dN}{dt}$, the time rate of change of this amount of substance, is proportional to amount of substance present, then $\frac{dN}{dt} = kN$ or $\frac{dN}{dt} - kN = 0$, where k is the constant of proportionality.

(ii) **Temperature problem** Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of temperature of a body is proportional to the temperature difference between the body and its surrounding medium.

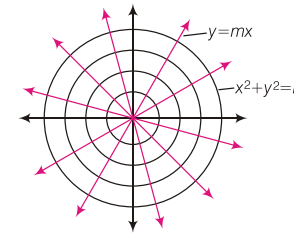
Let T and T_m respectively denote the temperatures of the body and medium, then time rate of change of temperature of the body is $\frac{dT}{dt}$ and Newton's law of cooling can be formulated as $\frac{dT}{dt} = -k(T - T_m)$ or $\frac{dT}{dt} + kT = kT_m$, where '$k$' is a positive constant of proportionality.

If $k > 0$ and $\frac{dT}{dt} < 0 \Rightarrow$ Cooling process

If $k > 0$ and $\frac{dT}{dt} > 0 \Rightarrow$ Heating process

Orthogonal Trajectories

Any curve, which cuts every member of given family of curves at right angle, is called an orthogonal trajectory of the family, e.g. Each straight line passing through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = r^2$.



Working Rule

Following steps are to be followed to find the orthogonal trajectory of given family of curves

Step I Let $f(x, y, c) = 0$ be the given equation, where c is an arbitrary parameter.

Step II Differentiate given equation with respect to x and eliminate c.

Step III Substitute $\left(\frac{-dx}{dy}\right)$ for $\left(\frac{-dy}{dx}\right)$ in the equation obtained in Step II.

Step IV Solve the differential equation obtained in Step III, to get required orthogonal trajectories.

Best Practice SHOTS

48. The population $p(t)$ at time t of a certain mouse species follows the differential equation $\frac{dp(t)}{dt} = 0.5\, p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is

(a) log 9 (b) $\frac{1}{2}$ log 18 (c) log 18 (d) 2 log 18

49. At present, a firm is manufacturing 2000 items. It is estimated the rate of change of production 'p' with respect to additional number of workers 'x' is given by $dp/dx = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is **[JEE Main 2013]**

(a) 2500 (b) 3000 (c) 3500 (d) 4500

50. Let the population of rabbits surviving at a time 't' be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$.

If $p(0) = 100$, then $p(t)$ is equal to **[JEE Main 2014]**

(a) $400 - 300\, e^{t/2}$ (b) $300 - 200\, e^{-t/2}$

(c) $600 - 500\, e^{t/2}$ (d) $400 - 300\, e^{-t/2}$

51. A body at a temperature is placed in a room which is hold at a constant temperature of 30°F. If after 10 min the temperature of the body is 0°F and after 20 min the temperature of the body is 15°F, then the initial temperature of the body is

(a) 30°F (b) – 30°F (c) 25°F (d) – 25°F

52. The orthogonal trajectories of the family of circles given by $x^2 + y^2 - 2ay = 0$ (a is the parameter), is

(a) $x^2 + y^2 - 2Kx = 0$ (b) $x^2 + y^2 - 2Ky = 0$

(c) $x^2 + y^2 - 2K_1x - 2K_2y = 0$ (d) None of these

Answers

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (a)
11. (a)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (b)
21. (c)	**22.** (a)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (d)	**30.** (b)
31. (b)	**32.** (d)	**33.** (c)	**34.** (d)	**35.** (a)	**36.** (a)	**37.** (a)	**38.** (c)	**39.** (d)	**40.** (c)
41. (a)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (a)	**46.** (a)	**47.** (c)	**48.** (d)	**49.** (c)	**50.** (a)
51. (b)	**52.** (a)								

Formulae at a Glance

1. Order of a differential equation is the order of the highest derivative. Degree of a differential equation is the power of the highest order derivative, when given equation is polynomial in derivatives. In a transcendental differential equation, we cannot determine the degree, i.e. degree is undefined.
2. Standard form of differential equation for variable separable method is $f_1(x)\,dx = f_2(y)\,dy$, where $f_1(x)$ and $f_2(y)$ are the functions in x alone and y alone respectively.
3. Standard form of homogeneous differential equations is $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$

   where, $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of same degree.

   For solution, put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ and if $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$, then put $x = vy$, $\frac{dx}{dy} = v + y\frac{dv}{dy}$.
4. Standard form of linear differential equation is $\frac{dy}{dx} + P\,y = Q$

   where, P and Q are functions of x only.

   Solution for this differential equation is given by

   $$y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C, \text{ where IF} = e^{\int P dx}.$$
5. Form of Bernoulli's differential equation is $\frac{dy}{dx} + Py = Qy^n$.

   First divide by y^n and then put $y^{1-n} = z$ and solve it.
6. *(i)* $d(x + y) = dx + dy$ *(ii)* $d(xy) = x\,dy + y\,dx$

(iii) $x\,dx + y\,dy = \frac{1}{2}d\,(x^2 + y^2)$

(iv) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$ *(v)* $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(vi) $d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2}$ *(vii)* $d\left(\frac{y^2}{x}\right) = \frac{2xydy - y^2dx}{x^2}$

(viii) $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(ix) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

(x) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

(xi) $d\,[\log(xy)] = \frac{x\,dy + y\,dx}{xy}$

(xii) $d\,[\log(x + y)] = \frac{dx + dy}{x + y}$

(xiii) $d\left\{\frac{1}{2}\ln(x^2 + y^2)\right\} = \frac{xdx + ydy}{x^2 + y^2}$

(xiv) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$

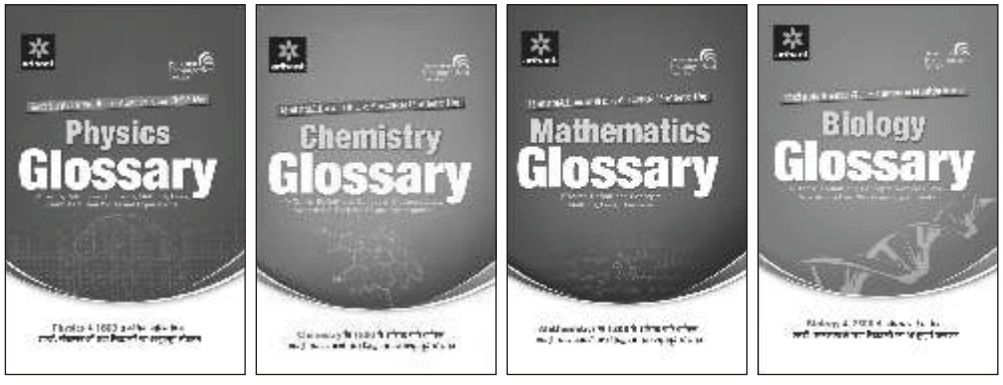
(xv) $d\left(\frac{e^y}{x}\right) = \frac{xe^ydy - e^ydx}{x^2}$

(xvi) $d\left(\frac{e^x}{y}\right) = \frac{ye^xdx - e^xdy}{y^2}$

(xvii) $d(x^m\,y^n) = x^{m-1}\,y^{n-1}\,(my\,dx + nx\,dy)$

GLOSSARIES of
(Physics, Chemistry, Mathematics & Biology)

arihant



10+2, इंजीनियरिंग व मेडिकल
प्रवेश परीक्षाओं में अध्ययनरत् विद्यार्थियों के लिए पारिभाषिक शब्दों, संकल्पनाओं तथा सिद्धान्तों का अभूतपूर्व संकलन

Matrices and Determinants

Matrices and Determinants are important topics for JEE Main and Advanced. Atleast two questions are asked every year from these topics. As we thought about student convenience, we are dividing this topic into five sub-topics, *viz.* 'Matrix and its properties', 'Some special Matrices and inverse of a matrix', 'Solution of system of at linear equations by matrix method as well as rank method', 'Determinant, its properties and solution of system of linear equations by determinants' and Some important terms related to determinants'.

MATRIX AND ITS PROPERTIES

Matrix

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as $m \times n$ matrix. Such an array is enclosed by [ ] or ( ). An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m \times n}$. The numbers $a_{11}, a_{12}, \dots,$ etc., are known as elements of the matrix A, a_{ij} belong to the ith row and jth column and is called (i, j)th element of the matrix $A = [a_{ij}]$.

Classification of Matrices

1. **Row matrix** A matrix having only one row, is called a row matrix or row vector.
2. **Column matrix** A matrix having only one column, is called a column matrix or a column vector.
3. **Horizontal matrix** If in a matrix, the number of rows is less than the number of columns, then it is called a horizontal matrix.
4. **Vertical matrix** If in a matrix, the number of rows is greater than number of columns, then it is called the vertical matrix.
5. **Zero matrix or null matrix** A matrix each of whose elements are zero, is called zero matrix or a null matrix.
6. **Square matrix** A matrix in which number of rows is equal to the number of columns, say m, is called square matrix of order m.
7. **Diagonal matrix** A square matrix $A = [a_{ij}]_{n \times n}$ is called diagonal matrix, if all the elements except in the leading diagonal are zero, i.e. $a_{ij} = 0$ for $i \neq j$. It is written as $A = \text{diag}\,(a_{11}, a_{22}, \dots, a_{nn})$.
8. **Scalar matrix** A square matrix in which every non-diagonal elements are zero and all diagonal elements are equal, is known as a scalar matrix.
9. **Unit matrix** A square matrix in which every non-diagonal elements are zero and every diagonal elements are 1, is called a unit matrix or an identity matrix. Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is unit matrix, if $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ 1, & \text{when } i = j \end{cases}$

   A unit matrix of order n is denoted by I_n or I.
10. **Upper triangular matrix** A square matrix $A = [a_{ij}]$ is called upper triangular matrix, if $a_{ij} = 0, \forall\, i > j$.

    e.g. The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix.
11. **Lower triangular matrix** A square matrix $A = [a_{ij}]$ is called a lower triangular matrix, if $a_{ij} = 0, \forall\, i < j$.

    e.g. The matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 4 & 6 \end{bmatrix}$ is a lower triangular matrix.
12. **Trace of a matrix** The sum of the diagonal elements of a square matrix A is called the trace of A and is denoted by tr (A).

    e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then tr $(A) = 1 + 5 + 9 = 15$
13. **Submatrix** A matrix, which is obtained from a given matrix by deleting any number of rows or columns or both, is called submatrix of given matrix.
14. **Equality of matrices** Two matrices A and B are said to be equal, if they are of same order and all corresponding elements are equal. It is written as $A = B$.

REMEMBER

- No element of principal diagonal in diagonal matrix is zero.
- Number of zeroes in diagonal matrix is given by $n^2 - n$, where n is the order of the matrix.
- A diagonal matrix of order $n \times n$ having $d_1, d_2, \dots, d_n$ as diagonal elements, is denoted by diag $(d_1, d_2, \dots, d_n)$.
- A matrix is said to be triangular, if it is either upper triangular or lower triangular.
- Diagonal matrix is both upper and lower triangular.
- Multiplication of two diagonal matrices is also a diagonal matrix and diag $(a_1, a_2, \dots, a_n) \times$ diag $(b_1, b_2, \dots, b_n)$
  $= \text{diag}\,(a_1 b_1, a_2 b_2, a_3 b_3, \dots, a_n b_n)$
- The multiplication of two scalar matrices is also a scalar matrix.
- Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of matrix.
- A triangular matrix $A = [a_{ij}]_{n \times n}$ is called strictly triangular, if $a_{ij} = 0$ for $1 \leq i \leq n$.
- The multiplication of two triangular matrices is a triangular matrix.

Addition of Matrices

Let A and B be two matrices. Then, the sum of matrices defined only, if A and B are of same order. Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of same order, then the sum $A + B$ is defined and $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

Properties of Addition of Matrices

(i) **Matrix addition is commutative** If A and B are two matrices of the same order, then $A + B = B + A$.

(ii) **Matrix addition is associative** If A, B and C are three matrices of same order, then $(A + B) + C = A + (B + C)$.

(iii) **Existence of additive identity** If O is null matrix of same order as that of matrix A, then $A + O = O + A = A$.

(iv) **Existence of additive inverse** If A is any matrix, then $A + (-A) = O = (-A) + A$.

(v) **Cancellation laws** If A, B and C are matrices of same order, then

$A + B = A + C \Rightarrow B = C$ [left cancellation law]

$B + A = C + A \Rightarrow B = C$ [right cancellation law]

Subtraction of Matrices

Let A and B be two matrices of the same order, then subtraction of two matrices $(A - B)$ is basically sum of matrix A with additive inverse of the matrix B i.e. $(-B)$. We can obtained the elements of $A - B$ by subtracting the elements of B from A.

Multiplication of Matrix by Scalar

Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar. The matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus, if $A = [a_{ij}]_{m\times n}$, then $kA = [ka_{ij}]_{m\times n}$.

Multiplication of Matrices

Two matrices A and B can be multiplied only, if number of columns in A is same as the number of rows in B.

e.g. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$, respectively. Then, their product AB is of order $m \times p$.

$$\underbrace{A_{m\times n}}_{\text{Number of columns of } A \text{ = Number of rows of } B} \cdot B_{n\times p} = \underbrace{AB_{m\times p}}_{\text{Order of resultant matrix}}$$

Properties of Matrix Multiplication

(i) **Multiplication is distributive over matrix addition** If A, B and C are the matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively, then $A(B + C) = AB + AC$.

(ii) **Multiplication is associative** If A, B and C are matrices of order $m \times n$, $n \times p$ and $p \times r$ respectively, then $(AB)C = A(BC)$.

(iii) **Multiplicative identity** If A is an $m \times n$ matrix, I_n is the identity matrix of order $n \times n$ and I_m is the identity matrix of order $m \times m$, then $I_m A = A$ and $AI_n = A$. In particular, A is a square matrix of order n, then $AI_n = I_n A = A$.

REMEMBER

- If A and B are two square matrices of the same order, say n, then both the products AB and BA are defined and each is a square matrix of order n.
- In the matrix product AB, the matrix A is called pre-multiplier (pre-factor) and B is called post-multiplier (post-factor).
- The rule of multiplication of matrices is row-columnwise $(\rightarrow \downarrow)$, i.e. the first row of AB is obtained by multiplying the first row of A with first, second, third,... columns of B, respectively. Similarly, second row of AB is obtained by multiplying the second row of A with first, second, third,... columns of B respectively and so on.
- $AB = O$ (null matrix) does not necessarily imply that $A = O$ or $B = O$ or $A, B = O$.
  e.g. If $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq O$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- If A is a square matrix of order n, then A^2 is defined as AA. In general, $A^m = A \cdot A \ldots A$ (m times), where m is any positive integer.
- If I is the unit matrix, then $I = I^2 = I^3 = \ldots = I^n$.

Transpose of a Matrix

Let A be an $m \times n$ matrix. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A and is denoted by A' or A^T or A^C, thus

(i) If order of A is $m \times n$, then order of A^T is $n \times m$.

(ii) (i, j)th element of A = (j, i)th element of A^T.

Properties of Transpose of a Matrix

(i) $(A^T)^T = A$ (ii) $(A + B)^T = A^T + B^T$ (iii) $(AB)^T = B^T A^T$

(iv) $(kA)^T = kA^T$ (v) $(A^n)^T = (A^T)^n$

(vi) Trace (A^T) = Trace (A) (vii) Trace $(AA^T) \geq 0$

Conjugate of a Matrix

The matrix obtained from any given matrix A by taking conjugate of its each elements is called conjugate of A and is denoted by $\overline{A}$.

Properties of Conjugate

(i) If A is any matrix, then $(\overline{A}) = A$.

(ii) If A and B are matrices of same order, then
$$(\overline{A + B}) = \overline{A} + \overline{B}.$$

(iii) If A is any $m \times n$ and B is any $n \times p$ matrix, then
$$(\overline{AB}) = \overline{A} \cdot \overline{B}.$$

(iv) If A is any matrix and k is any scalar, then
$$(\overline{kA}) = k\,\overline{A}.$$

(v) If A is any square matrix, then $(\overline{A^n}) = (\overline{A})^n$.

Transpose Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called transpose conjugate of A and is denoted by A^θ or A^*.

Properties of Transpose Conjugate

(i) $(\overline{A^\theta}) = (\overline{A})' = A^\theta$ (ii) $(A^\theta)^\theta = A$ (iii) $(A + B)^\theta = A^\theta + B^\theta$

(iv) $(k\,A)^\theta = \overline{k}\,A^\theta$, k being any number

(v) $(AB)^\theta = B^\theta A^\theta$ (vi) $(A^n)^\theta = (A^\theta)^n$

Best Practice SHOTS

1. If $\begin{bmatrix} x & 0 \\ 1 & y \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix}$, then [RPET 1994]

(a) $x = -3, y = -2$ (b) $x = 3, y = -2$
(c) $x = 3, y = 2$ (d) $x = -3, y = 2$

2. If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is commutative with the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then [WB JEE 2008]

(a) $a = 0, b = c$ (b) $b = 0, c = d$
(c) $c = 0, d = a$ (d) $d = 0, a = b$

3. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then the value of X^n is

(a) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$
(c) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (d) None of these

4. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $A_\alpha A_\beta$ is equal to

(a) $A_{\alpha+\beta}$
(b) $A_{\alpha\beta}$
(c) $A_{\alpha-\beta}$
(d) None of the above

5. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A+I)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, the value of $(a+b+c+d)$ is

(a) 2
(b) 1
(c) 4
(d) None of the above

6. If $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$, $Q = PP^T$, then the value of the determinant of Q is [WB JEE 2012]

(a) 2 (b) –2
(c) 1 (d) 0

7. If $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P(Q^{2005})P^T$ is equal to [IIT Screening 2005]

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{3}/2 & 2005 \\ 1 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 2005 \\ \sqrt{3}/2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & \sqrt{3}/2 \\ 0 & 2005 \end{bmatrix}$

8. Let $P = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix}$ and $X = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Then, P^3X is equal to [WB JEE 2013]

(a) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
(c) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

9. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $A^TA = AA^T = I$, then xy is equal to

(a) –1 (b) 1 (c) 2 (d) –2

10. If A and B are square matrices such that $A^* = A$ and $B^* = B$, where A^* denotes the conjugate transpose of A, then $(AB - BA)^*$ is equal to

(a) null matrix
(b) $AB - BA$
(c) $BA - AB$
(d) None of the above

MATHEMATICS IN action!

New Biggest Prime Number

The largest known prime number, newly discovered, is almost five million digits longer than the previous record-holder.

In a computer laboratory, a computer multiplied 74,207,281 two's together and subtracted 1. It then checked that this number was not divisible by any positive integer except 1 and itself; the definition of a prime number.

This immense number can only be practically written down in mathematical notation using exponents: $2^{74,207,281} - 1$.

The previous largest was $2^{57,885,161} - 1$, which has a mere 17 million or so digits.

This is the 15th prime number found by the Great Internet Mersenne prime Search or GIMPS. Mersenne primes are those that can be written in the form $2^n - 1$ where n is an integer. They are named after Marin Mersenne, a French theologian and mathematician who studies them in the early 17th century.

SOME SPECIAL MATRICES AND INVERSE OF A MATRIX

Some Special Matrices

Following are some special types of matrices

Symmetric Matrix

A square matrix A is said to be symmetric, if $A' = A$, i.e. the matrix $A = [a_{ij}]_{n \times n}$ is symmetric, if $a_{ij} = a_{ji}$ for all i and j.

Skew-symmetric Matrix

A square matrix A is said to be skew-symmetric, if $A^T = -A$

i.e. the matrix $A = [a_{ij}]_{m \times n}$ is skew-symmetric,

if $a_{ij} = -a_{ji}$ for all i and j.

Note Elements of main diagonal of a skew-symmetric matrix are all zero, because by definition, $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$ for all values of i.

SPEED **BOOSTER**

- If A is a square matrix, then
  (a) $A + A^T$ is symmetric matrix.
  (b) $A - A^T$ is skew-symmetric matrix.
- If A and B are two symmetric (or skew-symmetric) matrices of same order, then $A + B$ is symmetric (or skew-symmetric).
- Scalar multiple of symmetric (or skew-symmetric) matrix is also symmetric (or skew-symmetric) matrix.
- Product of two symmetric matrices A and B is symmetric matrix, if $AB = BA$.
- Every square matrix can be expressed uniquely as sum of symmetric and skew-symmetric matrices.
- The matrix $B^T AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric matrix.
- All positive integral powers of symmetric matrix are symmetric.
- All positive odd powers of skew-symmetric matrix is skew-symmetric though, even powers of skew-symmetric matrix is symmetric.
- If A and B are two symmetric matrices of the same order, then
  (a) $AB - BA$ is a skew-symmetric matrix.
  (b) $AB + BA$ is a symmetric matrix.

Orthogonal Matrix

A square matrix of order $n \times n$ is said to be orthogonal, if $AA^T = I = A^T A$.

Properties of Orthogonal Matrix

(i) If A is an orthogonal matrix, then A^T is also orthogonal.

(ii) For any two orthogonal matrices A and B, AB and BA are orthogonal matrices.

(iii) If A is an orthogonal matrix, then A^{-1} is also orthogonal.

Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be hermitian matrix, if $a_{ij} = \overline{a_{ji}}$, $\forall\, i, j$, i.e. $A = A^\theta$.

Skew-hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be a skew-hermitian matrix, if $a_{ij} = -\overline{a}_{ji}$, $\forall\, i, j$, i.e. $A^\theta = -A$.

Properties of Hermitian and Skew-hermitian Matrix

(i) If A and B are hermitian matrices of same order, then $AB + BA$ is hermitian and $AB - BA$ is skew-hermitian matrix.

(ii) Every square matrix can be uniquely expressed as $A + iB$, where A and B are hermitian matrices.

(iii) Every square matrix can be uniquely represented as the sum of a hermitian and a skew-hermitian matrices.

(iv) If A is a square matrix, then $A + A^*$ is hermitian and $A - A^*$ is skew-hermitian matrix.

(v) If A is a skew-hermitian matrix, then iA is hermitian and $\overline{A}$ is skew-hermitian matrix.

Singular Matrix

A square matrix A is said to be singular matrix, if determinant of A denoted by det (A) or $|A|$ is zero, i.e. if $|A| = 0$, then it is singular matrix, otherwise it is called non-singular matrix.

Note If $A = \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix}$, then $|A| = a(fj - ig) - b(ej - hg) + c(ei - hf)$

Idempotent Matrix

A square matrix A is said to be idempotent, if $A^2 = A$.

Properties of Idempotent Matrix

(i) If A and B are idempotent matrices, then AB is an idempotent matrix iff $AB = BA$.

(ii) If A and B are idempotent matrices, then $A + B$ is an idempotent iff $AB = BA = O$.

(iii) If A is an idempotent and $A + B = I$, then B is an idempotent and $AB = BA = O$.

(iv) Diagonal (1, 1, 1 ..., 1) is an idempotent matrix.

(v) If $AB = A$ and $BA = B$, then $A^2 = A, B^2 = B$.

Involuntary Matrix

A square matrix A is said to be involuntary matrix, if $A^2 = I$.

Nilpotent Matrix

A square matrix A is said to be nilpotent matrix, if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and C_{ij} be cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A, denoted by adj (A), is defined as the transpose of the cofactor matrix. The adjoint of a square matrix A is obtained on replacing each (i, j)th element of A by the cofactor of (j, i)th element in A.

SPEED BOOSTER

- If A is a square matrix of order n, then $A(\text{adj } A) = |A| I_n = (\text{adj } A)A$, where I_n is a square matrix of order n.
- If A is square matrix, then adj $(A') = (\text{adj } A)'$.
- If A and B are two square matrices of the same order, then adj $(AB) = (\text{adj } B)(\text{adj } A)$.
- adj $(\text{adj } A) = |A|^{n-2} A$, where A is a non-singular square matrix of order n.
- $|A \cdot \text{adj } A| = |A|^n$, where A is square matrix of order n.
- $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$, where A is a non-singular square matrix of order n.

Inverse of a Square Matrix

Let A be any n-rowed square matrix. Then, a square matrix B, such that $AB = BA = I$ is called inverse of A. The inverse of A is denoted by A^{-1} and is determined by the formula $A^{-1} = \frac{1}{|A|}(\text{adj } A)$ Provided $|A| \neq 0$. It may be noted that $AA^{-1} = A^{-1}A = I$.

Elementary Transformations

Each one of the following operations on a matrix is called an elementary transformation.

(i) Interchanging any two rows or columns.
This transformation is indicated by $R_i \to R_j$ $(C_i \to C_j)$.

(ii) Multiplication of elements of any row or column by a non-zero scalar quantity k. This transformation is indicated by $R_i \to kR_i (C_i \to kC_i)$.

(iii) Addition of constant multiple k $(k \neq 0)$ of the elements of any row or column to the corresponding element of any other row or column. This transformation is indicated by $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Note Two matrices are said to be equivalent, if one is obtained from the other by elementary transformations. The symbol '$\approx$' is used for equivalence.

REMEMBER

- A square matrix is invertible if and only if it is non-singular matrix.
- The inverse of the inverse matrix is the original matrix itself, i.e. $(A^{-1})^{-1} = A$.
- The inverse of the transpose of matrix is the transpose of its inverse, i.e. $(A^T)^{-1} = (A^{-1})^T$.
- If A and B are two invertible matrices of the same order, then AB is also invertible and moreover $(AB)^{-1} = B^{-1}A^{-1}$.
- Let A, B and C be square matrices of the same order. If A is non-singular matrix, then
  $AB = AC \Rightarrow B = C$ [left cancellation law]
  $BA = CA \Rightarrow B = C$ [right cancellation law]
- If A is a non-singular matrix such that A is symmetric, then A^{-1} is also symmetric.
- If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$.

Best Practice SHOTS

11. If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then
(a) A is a symmetric matrix
(b) A is a skew-symmetric matrix
(c) A is an orthogonal matrix
(d) None of the above

12. If A is a symmetric matrix, then matrix $M'AM$ is
(a) symmetric (b) skew-symmetric
(c) hermitian (d) skew-hermitian

13. If A is a square matrix, then which of the following matrices is not symmetric?
(a) $A + A'$ (b) AA' (c) $A'A$ (d) $A - A'$

14. If $\begin{bmatrix} 2+x & 3 & 4 \\ 1 & -1 & 2 \\ x & 1 & -5 \end{bmatrix}$ is a singular matrix, then x is equal to [Kerala (Engg.) 2001; KCET 2007]
(a) 13 / 25 (b) –25 / 13 (c) 5 / 13 (d) 25/13

15. If Z is an idempotent matrix, then $(I + Z)^n$ is equal to
(a) $I + 2^n Z$ (b) $I + (2^n - 1)Z$
(c) $I - (2^n - 1)Z$ (d) None of these

16. Adjoint of the matrix, $N = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is
(a) N (b) $2N$
(c) $-N$ (d) None of these

17. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{6}[A^2 + cA + dI]$, where $c, d \in R$, then pair of values (c, d) is [IIT Screening 2005]
(a) (6,11) (b) (6, –11) (c) (–6, 11) (d) (–6, – 11)

18. A is a singular matrix, then A (adj A) is
(a) null matrix (b) I
(c) A (d) None of these

19. Which one of the following is correct?
(a) Skew-symmetric matrix of odd order is non-singular
(b) Skew-symmetric of odd order is singular
(c) Skew-symmetric matrix of even order is always singular
(d) None of the above

20. If $A(\theta) = \begin{bmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{bmatrix}$, then which of the following is not true? [IIT Screening 2004]

(a) $A(\theta)^{-1} = A(\pi - \theta)$
(b) $A(\theta) + A(\pi + \theta)$ is a null matrix
(c) $A(\theta)_{-1}$ is invertible for all $\theta \in R$
(d) $A(\theta)^{-1} = A(-\theta)$

21. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to [JEE Main 2013]

(a) 4 (b) 11 (c) 5 (d) 0

22. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then α is equal to [IIT Screening 2004]

(a) ± 3 (b) ± 2 (c) ± 5 (d) 0

23. If A is skew-symmetric and $B = (I - A)^{-1}(I + A)$, then B is

(a) singular (b) symmetric
(c) skew-symmetric (d) orthogonal

24. If A is non-singular and $(A-2I)(A-4I) = O$, then $\frac{1}{6}A + \frac{4}{3}A^{-1}$ is equal to

(a) I (b) O (c) $2I$ (d) $6I$

25. Let p be an odd prime number and T_p be the following set of 2×2 matrices

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots (p-1)\} \right\}.$$

The number of A in T_p such that A is either symmetric or skew-symmetric or both and det (A) divisible by p is [IIT JEE 2010]

(a) $(p-1)^2$ (b) $2(p-1)$
(c) $(p-1)^2 + 1$ (d) $2p - 1$

26. Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1 such that five of these entries are 1 and four are 0. The number of matrices in set A is [IIT JEE 2009]

(a) 12 (b) 6 (c) 9 (d) 3

27. Let $w \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ w & 1 & c \\ w^2 & w & 1 \end{bmatrix}$, where each of a, b and c is either w or w^2. Then, the number of distinct matrices in the set S is [IIT JEE 2011]

(a) 2 (b) 6 (c) 4 (d) 8

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD AS WELL AS RANK METHOD

Rank of a Matrix

A positive integer r is said to be the rank of a non-zero matrix A, if

(i) there exists atleast one minor in A of order r, which is not zero.
(ii) every minor in A of order greater than r is zero.

If rank of a matrix A is r, then it is written as $\rho(A) = r$.

Properties of Rank of a Matrix

(i) The rank of a zero matrix (null matrix) is defined as zero, i.e. $\rho(O) = 0$.
(ii) If I_n is unit matrix of order n. Then, rank of I_n is n, i.e. $\rho(I_n) = n$.
(iii) From the definition of the rank of a matrix, we conclude that
  (a) if matrix A does not possesses any non-zero minor of order r, then $\rho(A) < r$.
  (b) if atleast one minor of order r of the matrix is not equal to zero, then $\rho(A) \geq r$.
(iv) If every $(r+1)$th order minor of A is zero, then any higher order minor will also be zero.
(v) If A is $n \times n$ non-singular matrix, then $\rho(A) = n$.
(vi) Elementary transformation do not change the rank of a matrix.
(vii) If A' is a transpose of A, then $\rho(A) = \rho(A')$.
(viii) If A^* is transpose conjugate of A, then $\rho(A^*) = \rho(A)$.
(ix) $\rho(A + B) \leq \rho(A) + \rho(B)$
(x) If A and B are two matrices such that the product AB is defined, then rank of AB cannot exceed the rank of the either matrix.

Echelon Form of a Matrix

A non-zero matrix A is said to be in Echelon form, if A satisfies the following conditions

(i) All the non-zero rows, if any precede the zero rows.
(ii) The number of zeroes preceding the first non-zero element in a row is less than number of such zeroes in the succeeding row.
(iii) The first non-zero element in a row is unity.

Note The number of non-zero rows of a matrix given in the Echelon form is its rank.

Solution of a System of Linear Equations by Matrix Method

Consider a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

We can express these equations as a single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ or } A \cdot X = B$$

where, A = Coefficient matrix $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Now, if $|A| \neq 0$, then $X = A^{-1}B$ is the unique solution of $AX = B$.

Criterion of Consistency

Let $AX = B$ be a system of n linear equations in n variables.

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and $(\text{adj}\,A)B \neq O$, then system of equations is inconsistent, i.e. it has no solution.

(iii) If $|A| = 0$ and $(\text{adj}\,A)B = O$, then the system of equations may be either consistent or inconsistent. However, if the system of equations is consistent, then it will have infinite number of solutions.

Homogeneous System of Linear Equations

The system of equations $AX = B$ is said to be homogeneous, if the constants $b_1, b_2, \ldots, b_n$ are all zero, i.e. if the matrix B is a zero matrix. In this case system is of the form $AX = O$, where O is a null matrix of order $n \times 1$.

(i) If $|A| \neq 0$, then its only solution is $X = 0$, which is called the trivial or zero solution.

(ii) If $|A| = 0$, then $AX = O$ has a non-trivial solution. It will have infinitely many solutions.

Solution of Non-homogeneous System of Linear Equations by Rank Method

Let $AX = B$ be a system of n linear equations in n variables.

***Step* I** Write the augmented matrix $[A : B]$.

***Step* II** Reduce the augmented matrix to Echelon form using elementary row (column) operations.

***Step* III** Determine the rank of coefficient matrix A and augmented matrix $[A : B]$ by counting the number of non-zero rows in echelon form of A and $[A : B]$.

***Step* IV**

(i) If $\rho(A) \neq \rho[A : B]$, then the system of equations is inconsistent.

(ii) If $\rho(A) = \rho[A : B]$ = number of unknowns. Then, the system of equation is consistent and has a unique solution.

(iii) If $\rho(A) = \rho[A : B] <$ number of unknowns. Then, the system of equations is consistent and has infinitely many solutions.

Solution of Homogeneous System of Linear Equations by Rank Method

Let $AX = O$ be homogeneous system of linear equations.

(a) If $\rho(A)$ = Number of variables. Then, $AX = O$ has trivial solution only, i.e. $X = 0$.

(b) If $\rho(A) <$ Number of variables. Then, $AX = O$ has a non-trivial solution. It will have infinitely many solutions.

Best Practice SHOTS

28. If $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{bmatrix}$, then rank of A is [UPSEAT 2004]

(a) 0 (b) 1 (c) 2 (d) 3

29. The rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, is

(a) 4 (b) 3 (c) 2 (d) 1

30. The existence of the unique solution of the system of equations $x + y + z = \lambda$; $5x - y + \mu z = 10$; $2x + 3y - z = 6$ depends on

(a) only μ (b) only λ (c) λ and μ both (d) neither λ nor μ

31. The system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ $x + 2y + \lambda z = \mu$ has no solution for [Odisha JEE 2003; Delhi CEE 2009]

(a) $\lambda \neq 3, \mu = 10$ (b) $\lambda = 3, \mu \neq 10$

(c) $\lambda \neq 3, \mu \neq 10$ (d) None of these

32. If the trivial solution is the only solution of the system of equations $x - ky + z = 0$; $kx + 3y - kz = 0$; $3x + y - z = 0$, then the set of values of k is [AIEEE 2011]

(a) $R - \{2\}$ (b) $R - \{-3\}$ (c) $\{2, -3\}$ (d) $R - \{2, -3\}$

33. Let a, b, c be positive real numbers. The following system of equations in x, y and z

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has [IIT JEE 1995]

(a) no solution (b) unique solution

(c) infinitely many solutions (d) finitely many solutions

DETERMINANT, ITS PROPERTIES AND SOLUTION OF SYSTEM OF LINEAR EQUATIONS BY DETERMINANTS

Determinants

Every square matrix $A = [a_{ij}]$ of order n can be associated to a number (real or complex) which is called the determinant of the square matrix.

Second Order Determinant

A determinant of order two is of the form, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

and is equal to $a_{11}a_{22} - a_{12}a_{21}$.

Third Order Determinant

A determinant of order three is of the form, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

and is equal to

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}).$$

Minors and Cofactors

Minor Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n. Then, the minor M_{ij} of the element a_{ij} of the matrix A is the determinant of the square sub-matrix of order $(n-1)$ obtained by deleting ith row and jth column of the matrix A.

Cofactor Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n. Then the cofactor of the element a_{ij} of the matrix A is denoted by C_{ij} and is equal to $(-1)^{i+j}M_{ij}$, where M_{ij} is the minor of the element a_{ij} of the matrix A.

Properties of Determinants

Properties of determinant of order three only are stated below. However, these properties hold for determinants of any order. These properties help a good deal in the evaluation of determinants.

(i) The value of determinant remains unchanged, if rows are changed into columns and columns are changed into rows,

i.e. $|A| = |A^T|$

e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$

(ii) If any two rows or columns of a determinant are interchanged. Then, the value of determinant so obtained is the negative of the value of the original determinant.

e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(iii) If two rows or columns of a determinant are identical. Then, the value of determinant is zero.

e.g. $\begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} = 0$

(iv) If each element of a row or column of a determinant is multiplied by a constant k. Then, the value of the new determinant is k times the value of original determinant,

e.g. $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(v) If any two rows or columns of a determinant are proportional, then its value is zero.

e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$

(vi) If each element of a row or column of a determinant is the sum of two or more terms. Then, the determinant can be expressed as the sum of two or more determinants,

e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+c_1 & a_{22}+c_2 & a_{23}+c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_1 & c_2 & c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(vii) If each element of a row or column of a determinant is multiplied by a constant k and then added to the corresponding elements of some other row or column. Then, the value of determinant remains same,

e.g. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{21} & a_{32}+ka_{22} & a_{33}+ka_{23} \end{vmatrix}$

(viii) If each element of a row or column of a determinant is zero, then the value of determinant is zero.

(ix) If r rows (or r columns) become identical when a is substituted for x. Then, $(x-a)^{r-1}$ is factor of given determinant.

(x) The sum of product of elements of any row or column of a determinant with the cofactors of the corresponding elements of any other row or column is zero.

e.g. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ and so on.

(xi) The sum of product of elements of any row or column of a determinant (where, $|A|_{n\times n} = \Delta$) with the cofactors of the corresponding elements of same row or column gives value of determinant.

e.g. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

then $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta$

Note If $\Delta = |a_{ij}|$ is a determinant of order n, then value of the determinant $|A_{ij}|$, where A_{ij} is the cofactor of a_{ij}, is Δ^{n-1} provided $\Delta \neq 0$.

SPEED BOOSTER

Learn the following results to enhance the speed

- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$
- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$
- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)$
$\{(a^2+b^2+c^2)+(ab+bc+ca)\}$
- $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$
- $\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3(x+a+b+c+d)$
- $\begin{vmatrix} x^2 & (x+a)^2 & (x-a)^2 \\ y^2 & (y+a)^2 & (y-a)^2 \\ z^2 & (z+a)^2 & (z-a)^2 \end{vmatrix} = -4a^3(x-y)(y-z)(z-x)$

Solution of System of Linear Equations by Determinants

For Non-Homogeneous System of Linear Equations

1. Cramer's Rule (Solution of System of Linear Equations in Two Unknowns)

The solution of the system of equations, $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ is given by $x = \frac{D_1}{D}$ and $y = \frac{D_2}{D}$

where, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ provided $D \neq 0$.

2. Cramer's Rule (Solution of System of Linear Equations in Three Unknowns)

The solution of the system of equations,

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$

and $a_3x + b_3y + c_3z = d_3$ is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ provided $D \neq 0$.

Conditions for consistency The following cases may arise

(i) If $D \neq 0$, then the system is consistent and has a unique solution, which is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

(ii) If $D = 0$ and atleast one of the determinants D_1, D_2 and D_3 is non-zero. Then, the given system is inconsistent, i.e. it has no solution.

(iii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system may or may not be consistent. However, if it is consistent, then it has infinitely many solutions.

For Homogeneous System of Linear Equations

If the system of equation is homogeneous, then $D_1 = D_2 = D_3 = 0$ (value of determinant is zero, because one column has all elements = 0).

(i) If $D \neq 0$, then system has only trivial solution, i.e. $x = y = z = 0$.

(ii) If $D = 0$, then the system has infinitely many solutions.

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34. One of the roots of the given equation

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0 \text{ is}$$

[AMU 2012]

(a) $-(a+b)$ (b) $-(b+c)$ (c) $-a$ (d) $-(a+b+c)$

35. If $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then

k is equal to [Odisha JEE 2009]

(a) 1 (b) 2 (c) 3 (d) 4

36. The cofactor of the element '4' in the determinant $\begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 3 & 4 & 2 \\ 8 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}$ is

[MPPET 1987]

(a) 4 (b) 10 (c) –10 (d) –4

37. The value of $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$ is

(a) 0 (b) $\sin\alpha \sin\beta \sin\gamma$

(c) $\cos\alpha \cos\beta \cos\gamma$ (d) None of these

38. The values of x for which the matrix $\begin{bmatrix} -x & x & 2 \\ 2 & x & -x \\ x & -2 & -x \end{bmatrix}$ will be non-singular, are

[WB JEE 2008]

(a) $-2 \le x \le 2$ (b) for all x other than 2 and –2

(c) $x \ge 2$ (d) $x \le -2$

39. If $[\cdot]$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0, 0 \le y < 1, 1 \le z < 2$, then the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$, is

[Delhi CEE 1998]

(a) $[z]$ (b) $[y]$ (c) $[x]$ (d) None of these

40. If the system of equations

$ax + by + c = 0; bx + cy + a = 0; cx + ay + b = 0$ has infinitely many solutions, then the system of equations

$(b+c)x + (c+a)y + (a+b)z = 0$

$(c+a)x + (a+b)y + (b+c)z = 0$

$(a+b)x + (b+c)y + (c+a)z = 0$ has

(a) only one solution (b) no solution

(c) infinite number of solutions (d) None of these

41. The respective values of x, y, z on solving the system of linear equations $x + y + z = 6; x - y + z = 2$ and $3x + 2y - 4z = -5$ by Cramer's rule, are

(a) 1, 3, 2 (b) 3, 2, 1 (c) 1, 2, 3 (d) 2, 3, 1

SOME IMPORTANT TERMS RELATED TO DETERMINANTS

Characteristic Equation of a Matrix

The equation $|A - \lambda I_n| = 0$ is called characteristic equation of the matrix $A = [a_{ij}]_{n \times n}$. The values of λ are called eigen values of A.

(i) The sum of eigen values of A is equal to its trace.

(ii) The product of eigen values of matrix A is equal to its determinant.

(iii) The eigen values of an orthogonal matrix are of unit modulus.

(iv) The eigen values of a unitary matrix are of unit modulus.

(v) A and A' have same eigen values.

(vi) The eigen values of a skew-hermitian matrix are either purely imaginary or zero.

(vii) If λ is an eigen value of A, then $\bar{\lambda}$ is an eigen value of A^{θ}.

(viii) The eigen values of triangular matrix are its diagonal elements.

(ix) If λ is an eigen value of A and $|A| \neq 0$, then $1/\lambda$ is eigen value of A^{-1}.

(x) If λ is the eigen value of A and $|A| \neq 0$, then $|A|/\lambda$ is the eigen value of adj (A).

(xi) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of A, then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$.

Product of Determinants of Same Order

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then, row by row multiplication of Δ_1 and Δ_2 is given by

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column, column by row or column by column as required in the problem.

Differentiation of Determinants

Let $\Delta(x)=\begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ are functions of x.

Then, $\frac{d}{dx}\Delta(x)=\Delta'(x)=\begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$

Also, $\Delta'(x)=\begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row or column at a time, keeping others unchanged.

Integration of Determinants

If $\Delta(x)=\begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$,

then $\int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$

Here, $f(x)$ and $g(x)$ are two functions of x and λ_1, λ_2 are constants.

Note This formula is only applicable, if there is variable only in one row or column, otherwise expand the determinant and integrate.

Maximum and Minimum Values of Determinants When Elements are Known

If $|A|=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, where $a_i's \in \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$

Then, $|A|_{max}$, when diagonal elements are $\{\min(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)\}$ and non-diagonal elements are $\{\max(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)\}$.

Also, $|A|_{min} = -|A|_{max}$

Best Practice SHOTS

42. The value of $\begin{vmatrix} a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3 \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 \\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3 \end{vmatrix}$, is

(a) $a_1a_2a_3b_1b_2b_3$ (b) $x_1x_2x_3y_1y_2y_3$
(c) 0 (d) None of these

43. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then $\phi(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$ is a polynomial of degree

(a) 2 (b) 3 (c) 4 (d) None of these

44. If $\Delta(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then the value of $\frac{d^n}{dx^n}[\Delta(x)]$ at $x=0$ is

(a) -1 (b) 0 (c) 1 (d) dependent on a

45. If $D_k=\begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n} D_k = 56$, then n equals

(a) 4 (b) 6 (c) 8 (d) None of these

46. If $y=\sin mx$, then the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$, where $y_n=\frac{d^n y}{dx^n}$, is

(a) m^9 (b) m^2 (c) m^3 (d) None of these

47. Let $f(x)=\begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$, then the value of $\int_0^{\pi/2}[f(x)+f'(x)]dx$ is

(a) π (b) $\pi/2$ (c) 2π (d) $3\pi/2$

48. Let $\{D_1, D_2, D_3, \ldots, D_n\}$ be the set of third order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, \ldots, a_9$. Then,

(a) $\sum_{i}^{n} D_i = 1$ (b) $\sum_{i}^{n} D_i = 0$ (c) $D_i = D_j, \forall\, i, j$ (d) None

49. If $f(x)=\begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then the value of $\lim_{x\to 0}\frac{f(x)}{x}$ is

(a) 1 (b) -1 (c) zero (d) None of these

50. If $g(x)=\frac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree < 3, then

(a) $\int g(x)\,dx=\begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix}\div\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}+K$

(b) $\frac{dg(x)}{dx}=\begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix}\div\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

(c) $\int g(x)\,dx=\begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix}\div\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}+K$

(d) None of the above

Answers

1. (b)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (c)
11. (c)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (d)
21. (b)	**22.** (a)	**23.** (d)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (c)	**30.** (a)
31. (b)	**32.** (d)	**33.** (b)	**34.** (d)	**35.** (b)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (c)
41. (c)	**42.** (c)	**43.** (d)	**44.** (b)	**45.** (d)	**46.** (d)	**47.** (a)	**48.** (b)	**49.** (c)	**50.** (a)

Formulae at a Glance

1. If A, B and C are three matrices such that their product is defined, then $(AB)C = A(BC)$; $A(B + C) = AB + AC$

2. *(i)* If $AB = O$, it does not mean that $A = O$ or $B = O$
(ii) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

3. *(i)* $(A^T)^T = A$
(ii) $(A + B)^T = A^T + B^T$
(iii) $(kA)^T = kA^T$
(iv) $(AB)^T = B^T A^T$

4. If A is square matrix, then
(i) $A + A^T$, AA^T, $A^T A$ are symmetric matrices.
(ii) $A - A^T$ is skew-symmetric matrix.

5. If A, B are two symmetric matrices, then
(i) $A \pm B$, $AB + BA$ are also symmetric matrices.
(ii) $AB - BA$ is a skew-symmetric matrix.

6. If A, B are two skew-symmetric matrices, then
(i) $A \pm B$, $AB - BA$ are skew-symmetric matrices
(ii) $AB + BA$ is a symmetric matrix.

7. Every square matrix A can be uniquely expressed as sum of a symmetric and skew-symmetric matrix.

i.e. $$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

8. For a square matrix A, if $|A| = 0$, then it is singular matrix otherwise non-singular matrix.

9. Let A be a square matrix
(i) For orthogonal matrix, $AA^T = I$
(ii) For idempotent matrix, $A^2 = A$
(iii) For involuntary matrix, $A^2 = I$
(iv) For nilpotent matrix, $A^p = O$, $p \in N$

10. Let A be a square matrix of order n
(i) $A(\text{adj } A) = |A| I_n = (\text{adj } A)A$
(ii) $|\text{adj } A| = |A|^{n-1}$, if $|A| \neq 0$
(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} \cdot A$, if $|A| \neq 0$
(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-2)^2}$, if $|A| \neq 0$
(v) $\text{adj}(A^T) = (\text{adj } A)^T$
(vi) $\text{adj}(A^m) = (\text{adj } A)^m$, $m \in N$
(vii) $\text{adj}(kA) = k^{n-1}(\text{adj } A)$, $k \in R$

11. *(i)* $(A^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$
(ii) $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
(iii) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$
(iv) $A = \text{diag}(a_1, a_2, a_3, \ldots, a_n) \Rightarrow A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$
(v) If $|A| \neq 0$, then $A^{-1} = \dfrac{1}{|A|}\text{adj } A$

12. The sum of diagonal elements of a square matrix is the trace of matrix.

13. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

14. The rank of matrix in Echelon form is equal to the number of non-zero rows in that matrix.

15. The sum of the products of elements of any row (column) of a determinant with the cofactors of the corresponding elements of
(i) same row is
$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta$$
(ii) any other row is
$$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$$

16. According to Cramer's rule system of linear equations is
(i) consistent with unique solution
$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}, \text{ if } \Delta \neq 0$$
(ii) consistent with infinitely many solutions, if
$$\Delta = 0 = \Delta_1 = \Delta_2 = \Delta_3$$
(iii) inconsistent (no solution), if $\Delta = 0$ and atleast one of Δ_1, Δ_2 and Δ_3 is non-zero.

17. When system of equations is non-homogeneous, then according to matrix method, we have
(i) If $|A| \neq 0$, then the system is consistent and has a unique solution given by $X = A^{-1}B$.
(ii) If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system has no solution, i.e. system is inconsistent.
(iii) If $|A| = 0$ and $(\text{adj } A)B = O$, then the system may be consistent or inconsistent.
However, if the system of equations is consistent, then it will have infinite number of solution.

18. When system of equations is homogeneous, then according to matrix method, we have
(i) If $|A| \neq 0$, then the system has only trivial solution.
(ii) If $|A| = 0$, then the system has infinite number of (non-trivial) solutions.

19. $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

20. In a triangular matrix, the diagonal elements are the characteristic roots of the matrix.

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Questions to Measure Your Problem Solving Skills

1. In the argand plane, z_1, z_2 and z_3 are respectively the vertices of an isosceles ΔABC with $AC = BC$ and $\angle CAB = \theta$. If $I(z_4)$ is the incentre of triangle, then the value of $\left(\frac{AB}{IA}\right)^2 \left(\frac{AC}{AB}\right)$ is

(a) $\frac{(z_2 - z_1)(z_1 - z_3)}{(z_4 - z_1)^2}$ (b) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$

(c) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (d) $\frac{(z_2 + z_1)(z_3 + z_1)}{(z_4 + z_1)}$

2. The number of integral solutions of $\sum_{i=1}^{3} x_i = 24$, $\sum_{i=1}^{3} x_i^2 = 21$ and $\prod_{i=1}^{3} x_i = 440$ is

(a) 4 (b) 5 (c) 6 (d) 7

3. Given 'p' and 'q' as true and 'r' as false the truth values of $\sim p \wedge (q \vee \sim r)$ and $(p \to q) \wedge r$ are respectively

(a) T, F (b) F, F

(c) T, T (d) F, T

4. A fair coin is tossed 15 times. If the probability of getting head as many times in the first ten throws as in the last five is k, then $(32768)k$ is equal to

(a) 3003 (b) 3030

(c) 3330 (d) 330

5. If the equation of the curve of reflection of the ellipse $\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1$ about the line $x - y - 2 = 0$ is $16x^2 + 9y^2 + k_1 x - 36y + k_2 = 0$, then unit digit of $k_1 + k_2$ is

(a) 4 (b) 0

(c) 3 (d) 2

6. If $N_a = \{an : n \in N\}$, then $N_5 \cap N_7$ is equal to

(a) N_7 (b) N (c) N_{35} (d) N_5

7. If a function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are integers and $a > 0$ is such that $f\left(\sin\frac{\pi}{18}\right) = 0$.

Then, the smallest possible value of $f(1)$ is

(a) 1 (b) 2 (c) 3 (d) 4

8. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the unit vectors such that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$ and $\mathbf{a} \cdot \mathbf{c} = \frac{1}{2}$, then

(a) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar

(b) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$ are non-coplanar

(c) $\mathbf{b}$, $\mathbf{d}$ are non-parallel

(d) $\mathbf{a}$, $\mathbf{d}$ are parallel and $\mathbf{b}$, $\mathbf{c}$ are parallel

9. A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is

(a) $y = \frac{3}{\sqrt{7}}x + \frac{15}{\sqrt{7}}$ (b) $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$

(c) $y = 2\sqrt{\frac{3}{7}}x + 15\sqrt{7}$ (d) None of these

10. $ABCD$ is a rectangular field. A vertical lamp post of height 12 m stands at the corner A. If the angle of elevation of its top from B is $60°$ and from C is $45°$, then the area of the field is

(a) $48\sqrt{2}$ sq m (b) $48\sqrt{3}$ sq m

(c) 48 sq m (d) $12\sqrt{2}$ sq m

11. The bisector of the acute angle formed between the lines $4x - 3y + 7 = 0$ and $3x - 4y + 14 = 0$ has the equation

(a) $x + y + 3 = 0$ (b) $x - y - 3 = 0$

(c) $x - y + 3 = 0$ (d) $3x + y - 7 = 0$

12. The line which is parallel to X-axis and crosses the curve $y = \sqrt{x}$ at an angle $45°$, is

(a) $y = \frac{1}{4}$ (b) $y = \frac{1}{2}$

(c) $y = 1$ (d) $y = 4$

13. If two points $A(a, 0)$ and $B(-a, 0)$ are stationary and if $\angle A - \angle B = \theta$ in ΔABC, then the locus of C is
(a) $x^2 + y^2 + 2xy\tan\theta = a^2$ (b) $x^2 - y^2 + 2xy\tan\theta = a^2$
(c) $x^2 + y^2 + 2xy\cot\theta = a^2$ (d) $x^2 - y^2 + 2xy\cot\theta = a^2$

14. Let $f : R \to R$ be a differentiable function having $f(2) = 6$, $f'(2) = \frac{1}{48}$. Then, $\lim\limits_{x\to 2} \frac{\int_6^{f(x)} 4t^3\, dt}{x-2}$ is equal to
(a) 18 (b) 12 (c) 36 (d) 24

15. Equation of plane passing through line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and perpendicular to the plane $x + 2y + z = 12$ is given by $ax + by + cz + 4 = 0$, then
(a) $a = -8, b = 2, c = -5$ (b) $a = -9, b = -2, c = -5$
(c) $a = 9, b = -2, c = -5$ (d) None of these

16. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \ldots \binom{30}{20}\binom{30}{30}$ is equal to
(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$ (c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

17. The area (in sq unit) of the triangle formed by the points with polar coordinates $(1, 0)$, $\left(2, \frac{\pi}{3}\right)$ and $\left(3, \frac{2\pi}{3}\right)$ is
(a) $\frac{11\sqrt{3}}{4}$ (b) $\frac{5\sqrt{3}}{4}$ (c) $\frac{5}{4}$ (d) $\frac{11}{4}$

18. The sum of the series $1 + 3x + 6x^2 + 10x^3 + \ldots \infty$ will be
(a) $\frac{1}{(1-x)^2}$ (b) $\frac{1}{1-x}$ (c) $\frac{1}{(1+x)^2}$ (d) $\frac{1}{(1-x)^3}$

19. If AM of two numbers is twice of their GM, then the ratio of greatest number to smallest number is
(a) $7 - 4\sqrt{3}$ (b) $7 + 4\sqrt{3}$
(c) 21 (d) 5

20. The two circles $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$ touch each other
(a) externally (b) internally
(c) coincide (d) None of these

21. If $y(t)$ is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to
(a) $-\frac{1}{2}$ (b) $e + \frac{1}{2}$ (c) $e - \frac{1}{2}$ (d) $\frac{1}{2}$

22. Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then
(a) $\Delta_1 = -\Delta$ (b) $\Delta \neq \Delta_1$
(c) $\Delta - \Delta_1 = 0$ (d) None of these

23. If a_1, a_2, a_3 (with $a_1 > 0$) are in GP with common ratio r, then the value of r for which the inequality $9a_1 + 5a_3 > 14a_2$ holds, cannot be in the interval
(a) $\left[1, \frac{9}{5}\right]$ (b) $(-\infty, 0)$ (c) $\left[1, \frac{9}{2}\right]$ (d) $\left[\frac{5}{8}, 1\right]$

24. If $\int \frac{1}{(\sin x + 4)(\sin x - 1)}\, dx$
$= A\frac{1}{\tan\frac{x}{2} - 1} + B\tan^{-1}(f(x)) + C_1$, then
(a) $A = \frac{1}{5}, B = \frac{-2}{5\sqrt{15}}, f(x) = \frac{4\tan x + 3}{\sqrt{15}}$
(b) $A = -\frac{1}{5}, B = \frac{1}{\sqrt{15}}, f(x) = \frac{4\tan\left(\frac{x}{2}\right) + 1}{\sqrt{15}}$
(c) $A = \frac{2}{5}, B = -\frac{2}{5}, f(x) = \frac{4\tan x + 1}{5}$
(d) $A = \frac{2}{5}, B = \frac{-2}{5\sqrt{15}}, f(x) = \frac{4\tan\frac{x}{2} + 1}{\sqrt{15}}$

25. The value of a in order that $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x is given by
(a) $a \geq \sqrt{2}$ (b) $a < \sqrt{2}$ (c) $a \geq 1$ (d) $a < 1$

26. The value of $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \ldots + 2^{n-1}\tan(2^{n-1}\alpha) + 2^n\cot(2^n\alpha)$ is
(a) $\cot 2^n\alpha$ (b) $2^n\tan(2^n\alpha)$ (c) 0 (d) $\cot\alpha$

27. Let $f(x) = 1 + 3x^2 + 3^2x^4 + \ldots + 3^{30}x^{60}$. Then, $f(x)$ has
(a) atleast one maximum (b) exactly one maximum
(c) atleast one minimum (d) exactly one minimum

28. The value of $\int_a^b (x-a)^3(b-x)^4\, dx$ is
(a) $\frac{(b-a)^6}{82}$ (b) $\frac{(b-a)^7}{136}$ (c) $\frac{(1-a)^8}{280}$ (d) None of these

29. The mean of five observations is 4 and their variance is 5.2. If three of these observations are 1, 2 and 6. Then, the other two are
(a) 2 and 9 (b) 3 and 8 (c) 4 and 7 (d) 5 and 6

30. The number of real solutions of the equation $27^{1/x} + 12^{1/x} = 2 \cdot 8^{1/x}$ is
(a) one (b) two
(c) zero (d) infinite

Answers

1. (c)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (d)	**6.** (c)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (a)
11. (c)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (c)	**16.** (c)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (b)
21. (a)	**22.** (c)	**23.** (a)	**24.** (d)	**25.** (a)	**26.** (d)	**27.** (d)	**28.** (d)	**29.** (c)	**30.** (c)

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Questions to Measure Your Problem Solving Skills

Paper I

Integer Type

1. $\int \sin(101x)\cdot\sin^{99}x\,dx = \frac{\sin(100x)\sin x^{\lambda}}{\mu}$, then $\frac{\lambda}{\mu}$ is equal to

2. The value of $99^{50} - \frac{99}{1}\cdot 98^{50} + \frac{99\cdot 98}{1\cdot 2}\cdot 97^{50} - \ldots - \frac{99\cdot 98}{1\cdot 2}\cdot 2^{50} + 99$ is

3. The image of the point $(2-i)$ in the line $(1+i)z + (1-i)\bar{z} + 2 = 0$, is $x + iy$, then $|x+y|$ is

4. If $a^3 - 3a^2 + 5a - 17 = 0$ and $b^3 - 3b^2 + 5b + 11 = 0$ are such that $a + b$ is a real number, then the value of $a + b$ is

5. Let $f(x) = 2x^3 + ax^2 + bx$, $(a, b \in N)$, if graph of $f(x)$ cuts the X-axis at 3 distinct points, then the minimum value of $a + b$ is

6. $f : [0, 5] \to R$, $y = f(x)$ such that $f''(x) = f''(5-x), \forall\ x \in [0, 5]$ $f'(0) = 1$ and $f'(5) = 7$, then evaluate $\int_1^4 f'(x)dx - 4$.

7. If number of quadrilaterals formed by joining vertices of 20 sided polygon such that sides of quadrilateral do not coincide with sides of polygon are k, then number of prime factors of k is

8. If $c \in (2, 7)$ and $f : [2, 7] \to [0, \infty)$ be a continuous and differentiable function, then the value of $(f(7) - f(2))\cdot(f^2(7) + f^2(2) + f(7)\cdot f(2))$ is $\lambda f^2(c)f'(c)$. The value of $\frac{\lambda}{3}$ is

One or More than One Option Correct Type

9. A function $y = f(x)$ satisfying the differential equation $\sin x\frac{dy}{dx} - y\cos x + \frac{\sin^2 x}{x^2} = 0$ is such that $y \to 0$ as $x \to \infty$, then the statement which is correct?

(a) $\lim_{x\to 0} f(x) = 1$ *(b)* $\int_0^{\pi/2} f(x)dx$ is less than $\frac{\pi}{2}$

(c) $\int_0^{\pi/2} f(x)\,dx$ is greater than unity

(d) $f(x)$ is odd function

10. At a point P on the parabola $y^2 = 4ax$ tangent and normal are drawn. Tangent intersects the X-axis at Q and normal intersects the curve at R such that chord PR subtends 90° at its vertex, then

(a) $PQ = 2a\sqrt{6}$ *(b)* $PR = 6a\sqrt{3}$

(c) area of $\Delta PQR = 18\sqrt{2}a^2$ *(d)* $PQ = 3a\sqrt{2}$

11. If in a ΔABC, p, q, r are the altitudes from the vertices A, B, C to the opposite sides, then which of the following hold(s) good?

(a) $(\Sigma p)\left(\Sigma\frac{1}{p}\right) = (\Sigma a)\left(\Sigma\frac{1}{a}\right)$

(b) $(\Sigma p)(\Sigma a) = \left(\Sigma\frac{1}{p}\right)\left(\Sigma\frac{1}{a}\right)$

(c) $(\Sigma p)(\Sigma pq)(\Pi a) = (\Sigma a)(\Sigma ab)(\Pi p)$

(d) $\left(\Sigma\frac{1}{p}\right)\left(\Pi\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right)\right)(\Pi a^2) = 16R^2$

12. $f : R \to R$ is one-one, onto and differentiable function and graph of $y = f(x)$ is symmetrical about the point $(4, 0)$, then

(a) $f^{-1}(2010) + f^{-1}(-2010) = 8$

(b) $\int_{-2010}^{2018} f(x)\,dx = 0$

(c) If $f'(-100) > 0$, then root of $x^2 - f'(10)x - f'(10) = 0$ may be non-real

(d) If $f'(10) = 20$, then $f'(-2) = 20$

13. In a parallelogram $PQRS$ (vertices taken in order), if $P \equiv (4, -3)$, $Q \equiv (1, 2)$ and R lies on the parabola $y^2 + 4x = 0$, then which of the following points satisfy the locus of the point S?

(a) $\left(-\frac{13}{4}, 0\right)$ (b) $\left(0, \frac{-13}{4}\right)$

(c) $(0, -5 + 2\sqrt{3})$ (d) $(0, -5 - 2\sqrt{3})$

14. If A and B are two invertible matrices of the same order, then adj (AB) is equal to

(a) adj(B) adj(A) (b) $|B||A|\, B^{-1}A^{-1}$

(c) $|B||A|\, A^{-1}B^{-1}$ (d) $|A||B|\, (AB)^{-1}$

15. If θ_1, θ_2, θ_3 and θ_4 are four real numbers, then any root of the equation $\sin\theta_1 z^3 + \sin\theta_2 z^2 + \sin\theta_3 z + \sin\theta_4 = 3$ lying inside the unit circle $|z| = 1$, satisfying the inequality

(a) $|z| < \frac{2}{3}$ (b) $|z| > \frac{2}{3}$ (c) $|z| < \frac{1}{2}$ (d) $|z| > \frac{1}{2}$

16. The number of triangles that can be formed with the sides of length a, b and c, where a, b, c are integers such that $a \le b \le c$ is

(a) $\frac{1}{4}(c+1)^2$ when c is odd (b) $\frac{1}{2}c(c+1)^2$ when c is odd

(c) $\frac{1}{4}c(c+2)$ when c is even (d) $\frac{1}{4}c^2$ when c is even

17. Let the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be perpendicular and the unit vector $\hat{\mathbf{c}}$ be inclined at angle θ to both $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, if $\hat{\mathbf{c}} = x\hat{\mathbf{a}} + y\hat{\mathbf{b}} + z(\hat{\mathbf{a}} \times \hat{\mathbf{b}})$, then

(a) $x = y$ (b) $z^2 = 1 - 2x^2$

(c) $z^2 = -2\cos\theta$ (d) $y^2 = \frac{1 + \cos 2\theta}{2}$

18. Let $I_n = \int_0^1 x^n \tan^{-1} x\, dx$. If $a_n I_{n+2} + b_n I_n = c_n, \forall\ n \ge 1$, then

(a) $a_1, a_2, a_3, \ldots$ are in AP (b) $b_1, b_2, b_3, \ldots$ are in AP

(c) $c_1, c_2, c_3, \ldots$ are in HP

(d) $\frac{\pi}{2} - c_1, \frac{\pi}{2} - c_2, \frac{\pi}{2} - c_3, \ldots$ are in AP

Match the Column Type

19. Match the statements of Column I with the values of Column II.

	Column I	Column II
A.	A tangent to the parabola $x^2 + 4ay = 0$ cuts the parabola $x^2 = 4y$ at A and B. The locus of mid-point of AB is	p. $y^2(x + 2a) + 4a^3 = 0$
B.	Locus of mid-points of the chords of the hyperbola $x^2 - y^2 = k^2$ of slope a is	q. $x = ay$
C.	Locus of the points of intersection of tangents drawn at the ends of all normal chords of the parabola $y^2 = 4ax$ is	r. $(a + 2)x^2 = 4y$
D.	Locus of the foot of the perpendicular from the centre of the hyperbola $xy = a^2$ on a variable tangents is	s. $(x^2 + y^2)^2 = 4a^2xy$

20. Match the statements of Column I with the values of Column II.

	Column I	Column II
A.	The function $f(x) = \frac{e^{x\cos x} - 1 - x}{\sin x^2}$ is not defined at $x = 0$, the value of $f(0)$, so that f is continuous at $x = 0$, is	p. -1
B.	The value of the definite integral $\int_0^1 \frac{dx}{(x)^{1/2} + (x)^{1/3}}$ is equal to $a + b\log 2$, where a and b are integers, then $(a + b)$ is equal to	q. 0
C.	Given, $e^n \int_0^n \frac{\sec^2\theta - \tan\theta}{e^\theta} d\theta = 1$, then value of $\tan(n)$ is equal to	r. $\frac{1}{2}$
D.	Let $a_n = \int_{1/n+1}^{1/n} \tan^{-1}(nx)\, dx$ and $b_n = \int_{1/(n+1)}^{1/n} \sin^{-1}(nx)\, dx$, then the value of $\lim_{x\to\infty}\left(\frac{a_n}{b_n} - \frac{1}{2}\right)$ is	s. 1

Paper II

Integer Type

1. Two numbers are selected at random from the numbers $1, 2, \ldots n$. Let p denote the probability that the difference between the first and second is not less than m (where, $0 < m < n$). If $n = 25$ and $m = 10$, then the value of $5p$ is

2. If $x_1, x_2, x_3, \ldots, x_n$ be an AP with common difference $\frac{\pi}{6}$ and assume $\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \ldots + \sec x_{n-1} \sec x_n = k(\tan x_n - \tan x_1)$, then the value of k is

3. $\lim_{x\to 0}\left[\sum_{r=0}^{n}(-1)^r\, {}^nC_r\left(\sum_{k=0}^{n-r} {}^{n-r}C_k x^k 2^k\right)\cdot(x^2 - x)^r\right]^{1/x} = e^{n\lambda}$, then λ is equal to

4. If $\int_1^3 (\{x\})^{[x]} = \lambda$, where $\{\cdot\}$ and $[\cdot]$ denotes the fractional part and greatest integer functions respectively, then the value of 6λ is

5. A curve $y = f(x)$ is such that $f(x) \ge 0$ and $f(0) = 0$ and bounds a curvilinear trapezoid with base $[0, x]$ whose area is proportional to $(n + 1)$th power of $f(x)$. If $f(1) = 1$, then $\{f(4)\}^n$ is equal to

6. A variable straight line through the point of intersection of the lines $3x + 4y = 12$ and $4x + 3y = 12$ meets the coordinate axes in A and B. If locus of the mid-point of AB is $7xy = k(x + y)$, then the value of k is

7. Total number of solutions of $\sin\{x\} = \cos\{x\}$, where $\{x\}$ denotes the fractional part, in $[0, 2\pi]$ is equal to

8. If $|z - 1| + |z - 3| \le 8$, the range of value of $|z - 4|$ is $[\lambda, \mu]$, then the value of $|\lambda - \mu|$ is

One or More than One Option Correct Type

9. Which of the following functions have their range equal to R (the set of real numbers)?

(a) $x \sin x$

(b) $\frac{[x]}{\tan 2x}$, $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) - \{0\}$, where $[\cdot]$ denotes the greatest integer function

(c) $\frac{x}{\sin x}$

(d) $[x] + \sqrt{\{x\}}$, where $[\cdot]$ and $\{\cdot\}$ respectively denote the greatest integer function and fractional part function.

10. In a ΔABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2(A/2)$. If a, b and c denote the sides of the triangle opposite to the angles A, B and C respectively, then

(a) $b + c = 4a$ (b) $b + c = 2a$

(c) locus of point A is an ellipse

(d) locus of point A is a pair of straight lines

11. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi)^x \sin x}\, dx$, $n = 0, 1, 2, 3, \ldots$, then

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$ (c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

12. Consider the family of planes $x + y + z = c$, where c is a parameter, intersecting the coordinate axes at P, Q and R and α, β and γ are the angles made by each member of this family with positive X, Y and Z-axes, which of the following interpretations hold good for this family?

(a) Each member of this family is equally inclined with the coordinate axes

(b) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$

(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$

(d) For $c = 3$, area of the ΔPQR is $3\sqrt{3}$ sq units

13. If the roots of the equation $x^3 + ax^2 + bx - 1 = 0$ are in increasing GP, then

(a) $a - 2b = 0$ (b) $b \in (3, \infty)$ (c) one root is 2

(d) one root is greater than one and other root is less than one

14. Tangents are drawn from the origin to the curve $y = \sin 2x$. The point of contact lie on

(a) $y^2 = 4x^2(1 - y^2)$ (b) $x^2 - y^2 - x^2y^2 = 0$

(c) $x^2 + y^2 = 2x^2y^2$ (d) $\frac{1}{1-y^2} = \sec^2 2x$

15. f is a continuous function in $[a, b]$, g is a continuous function in $[b, c]$. A function $h(x)$ is defined as

$h(x) = \begin{cases} f(x) & \text{for } x \in [a, b) \\ g(x) & \text{for } x \in (b, c] \end{cases}$, if $f(b) = g(b)$, then

(a) $h(x)$ has a removable discontinuity at $x = b$

(b) $h(x)$ may or may not be continuous in $[a, c]$

(c) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$

(d) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

16. $2\tan^{-1}(-3)$ is equal to

(a) $-\cos^{-1}\left(\frac{-4}{5}\right)$ (b) $-\pi + \cos^{-1}\left(\frac{4}{5}\right)$

(c) $-\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right)$ (d) $\cot^{-1}\left(\frac{4}{3}\right)$

Linked Comprehension Type

Passage I (Q. Nos. 17-18) *$f(x)$ is a continuous function satisfying $x^4 - 4x^2 \le f(x) \le 2x^2 - x^3$ for all $x \in [0, 2]$, such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, the Y-axis and the line $x = t$ $(0 \le t \le 2)$ is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, Y-axis and the line $x = t$ $(0 \le t \le 2)$.*

17. The function $f(x)$ is given by

(a) $\frac{1}{k+1}[x^4 - kx^3 + 2(k-2)x^2]$

(b) $\frac{1}{k+1}[x^4 - (k-2)x^3 + 2(k-2)x^2]$

(c) $\frac{1}{k+1}[x^4 - (k-2)x^3 + 2(k-2)x]$

(d) $\frac{1}{k+1}[x^4 - kx^3 + 2(k-2)x]$

18. If $k = 0$, then $f(x)$ is symmetric about the

(a) origin (b) Y-axis (c) X-axis (d) curve $y = x$

Passage II (Q. Nos. 19-20) *If n is a positive integer and x is any real number, then*

$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$,

where ${}^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$ are binomial coefficient.

19. The value of ${}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \ldots + {}^{950}C_0$ is

(a) ${}^{1001}C_{50}$ (b) ${}^{1002}C_{51} - {}^{1001}C_{51}$

(c) ${}^{1001}C_{951}$ (d) ${}^{1002}C_{51} - {}^{1001}C_{950}$

20. If $1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^n = \sum_{k=0}^{n} a_k x^k$, then

(a) $\sum_{k=0}^{n} a_k = 2^{n+1} - 1$ (b) $a_{n-2} = \frac{n(n+1)}{2}$

(c) $a_p > a_{p-1}$ for $p < \frac{n}{2}$

(d) $(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10}({}^{n+1}C_{10} - {}^{n+1}C_9)$

Answers

Paper I

1. (1) **2.** (0) **3.** (1) **4.** (2) **5.** (4) **6.** (8) **7.** (3) **8.** (5) **9.** (a,b,c) **10.** (a,b,c)
11. (a, c, d) **12.** (a,b,d) **13.** (a,c,d) **14.** (a,b,d) **15.** (b,d) **16.** (a,c) **17.** (a,b,c,d) **18.** (a,b,d)
19. A → r, B → q, C → p, D → s **20.** A → r, B → p, C → s, D → q

Paper II

1. (1) **2.** (2) **3.** (3) **4.** (5) **5.** (4) **6.** (6) **7.** (7) **8.** (8) **9.** (a,d) **10.** (b,c)
11. (a,b,c) **12.** (a,b,c) **13.** (b,d) **14.** (a,d) **15.** (a,c) **16.** (a,b,c) **17.** (a) **18.** (b) **19.** (a,b,c,d) **20.** (a,b,c,d)

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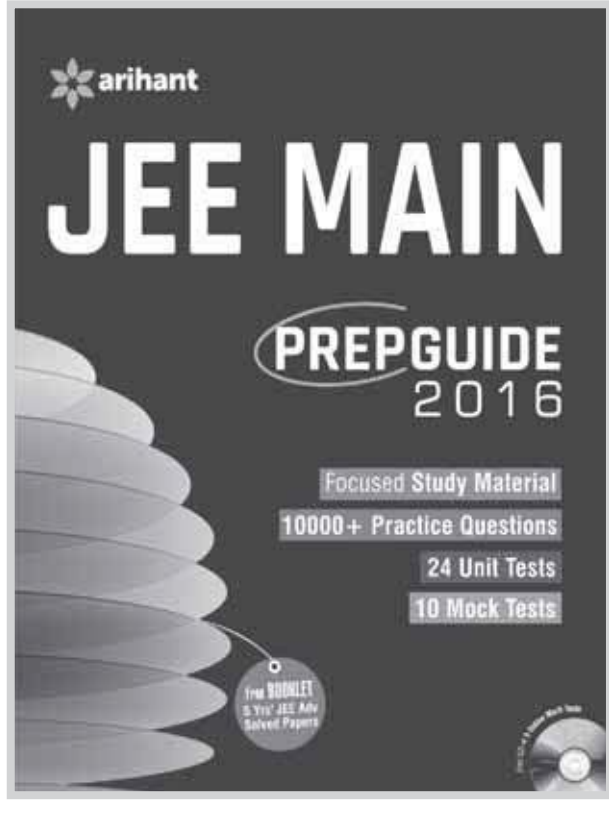

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MATHEMATICS SPECTRUM *for You*

In this section, we invite our readers to send their problems and we will try to provide the best possible solutions given by our expert panel.

Q1 Hi, I am a regular reader of your Mathematics spectrum and please post my following problem on the section 'Mathematics spectrum for you'.

Suppose that m and n are integers such that both the quadratic equations $x^2 - mx + n = 0$ and $x^2 + mx - n = 0$ have integer roots. Prove that n is divisible by 6.

Give solution and explanation.

[Sanmitra Chhatterjee, Junagarh]

➥ Dear Sanmitra, we know that, if a quadratic equation of the form $x^2 + bx + c = 0$, where $b, c \in I$, have integral roots, then discriminant (D) is a perfect square of integers. Let a and b be any integers such that

$$m^2 - 4n = a^2$$

and $$m^2 + 4n = b^2 \quad \ldots(i)$$

$\therefore$ $$8n = b^2 - a^2$$

and $$2m^2 = a^2 + b^2$$

Case I If m is divisible by 3, then a and b both are divisible by 3. Therefore, n is divisible by 3.

Case II If m is not divisible by 3, then a and b both are not divisible by 3. Therefore, $(b^2 - a^2)$ is divisible by 3 and hence n is divisible by 3.

Case III If m is odd, then a is also odd and therefore $4n = m^2 - a^2$ is divisible by 8. So, n is even.

Case IV If m is even, then both a and b are even. Now, from Eq. (i), we have

$$\left(\frac{m}{2}\right)^2 - n = \left(\frac{a}{2}\right)^2$$

and $$\left(\frac{m}{2}\right)^2 + n = \left(\frac{b}{2}\right)^2$$

$\therefore (b - a)/2$ is even. So, $n = (b^2 - a^2)/4$ is also even from all cases, it is proved that n is divisible by 6.

Q2 Sir, in the question given below, please explain why do we have to equate the domain in the RHS and that in LHS of the equation to arrive at the solution.

If f is an even function, then find the real values of x satisfying the equation $f(x) = f[(x + 1)/(x + 2)]$.

[Amit, Mumbai]

➥ Dear Amit, as we know that two functions are equal, if they are equal at each points of its domain. So, to define the equality of the functions, we need to equate their domains.

Given, $$f(x) = f\left(\frac{x+1}{x+2}\right) \quad \ldots(i)$$

and $f(x)$ is an even function.

$\therefore$ $$f(-x) = f(x) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we have

$$f(-x) = f(x) = f\left(\frac{x+1}{x+2}\right)$$

$\therefore$ $$-x = \frac{x+1}{x+2}$$

or $$x = \frac{x+1}{x+2}$$

$\Rightarrow$ $$-x^2 - 3x - 1 = 0$$

or $$x^2 + x - 1 = 0$$

$\Rightarrow$ $$x = \frac{-3 \pm \sqrt{5}}{2}$$

or $$x = \frac{-1 \pm \sqrt{5}}{2}$$

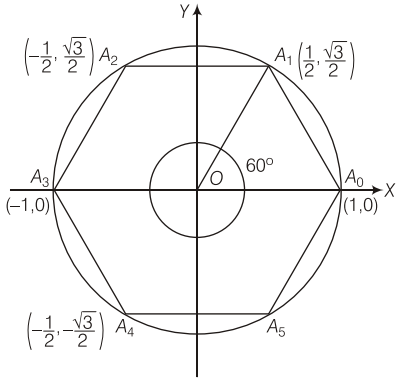
Thus, $$x = \left\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}\right\}$$

Q3 Sir, please provide me the solution of my following problem

Let $A_0, A_1, A_2, A_3, A_4, A_5$ be a regular hexagon inscribed in a unit circle with centre at the origin. Then, find the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4. **[Suchi, Panipat]**

➥ Dear Suchi, first of all read the question twice and draw a rough diagram according to the given information. Then, find the coordinates of A_0, A_1, A_2, A_3, A_4, A_5 and solve it.

Let O be the centre of the circle of unit radius and the coordinates of A_0 be (1, 0).



We know that coordinate of any point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.

Since, each side of the regular hexagon makes an angle of 60° at the centre 'O', then

Coordinates of $A_1 \equiv (\cos 60°, \sin 60°)$

$$= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Coordinates of $A_2 \equiv (\cos 120°, \sin 120°)$

$$= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Coordinates of $A_3 \equiv (-1, 0)$

Coordinates of $A_4 \equiv (\cos 240°, \sin 240°) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Coordinates of $A_5 \equiv (\cos 300°, \sin 300°) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Now, $A_0A_1 = \sqrt{\left(1-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$A_0A_2 = \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3} = A_0A_4$$

Hence, $(A_0A_1)(A_0A_2)(A_0A_4) = 1 \times \sqrt{3} \times \sqrt{3} = 3$

Q4 Sir, I am confused in finding the solution of the following problem. Please, provide me the best solution of it. If $x, y \in (0, 30)$ such that

$$\left[\frac{x}{3}\right] + \left[\frac{3x}{2}\right] + \left[\frac{y}{2}\right] + \left[\frac{3y}{4}\right] = \frac{11}{6}x + \frac{5}{4}y$$

(where, $[x]$ denotes greatest integer $\le x$), then find the number of ordered pairs (x, y). **[Jyoti, Chandigarh]**

➥ Dear Jyoti, first of all convert the greatest integer into fractional part by $\{x\} = x - [x]$, where $0 \le \{x\} < 1$, and then solve it.

Let $\{x\} = x - [x]$, where $\{x\}$ is the fractional part of x and $0 \le \{x\} < 1$

Now, we can write the given equation as

$$\frac{x}{3} - \left\{\frac{x}{3}\right\} + \frac{3x}{2} - \left\{\frac{3x}{2}\right\} + \frac{y}{2} - \left\{\frac{y}{2}\right\} + \frac{3y}{4} - \left\{\frac{3y}{4}\right\}$$

$$= \frac{11}{6}x + \frac{5}{4}y$$

$$\Rightarrow \quad \left\{\frac{x}{3}\right\} + \left\{\frac{3x}{2}\right\} + \left\{\frac{y}{2}\right\} + \left\{\frac{3y}{4}\right\} = 0$$

As each number on the LHS lies in the interval $0 \le x < 1$, we must have

$$\left\{\frac{x}{3}\right\} = \left\{\frac{3x}{2}\right\} = \left\{\frac{y}{2}\right\} = \left\{\frac{3y}{4}\right\} = 0$$

$\Rightarrow$ $\frac{x}{3}, \frac{3x}{2}, \frac{y}{2}$ and $\frac{3y}{4}$ must be integers.

$\therefore$ $x = 6, 12, 18, 24$ and $y = 4, 8, 12, 16, 20, 24, 28$

Hence, number of ordered pairs (x, y) equals to $4 \times 7 = 28$.

SHORTCUT Methods

*Here, we are introducing some important shortcut methods on **differential equation** to save your precious time in tedious calculations based problem, asked in various engineering entrance exams. So, learn these tricks and practice the questions based on each trick.*

1. If $y = Ae^{ax} + Be^{bx}$ is the solution of differential equation, where A and B are arbitrary constants, then corresponding differential equation is given by

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$$

Example The solution of a differential equation is $y = c_1e^{4x} + c_2e^{3x}$, the differential equation is given by

(a) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 7y = 0$ (b) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

(c) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$ (d) None of these

Sol. **(c)** Clearly, the required differential equation is

$$\frac{d^2y}{dx^2} - (4+3)\frac{dy}{dx} + 4\cdot 3\cdot y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$$

2. If $y = Ae^{ax} + Be^{bx} + Ce^{cx}$ is the solution of differential equation, where A, B and C are arbitrary constants, then corresponding differential equation is given by

$$\frac{d^3y}{dx^3} - (a+b+c)\frac{d^2y}{dx^2} + (ab+bc+ca)\frac{dy}{dx} - abcy = 0$$

Example If $y = Ae^{2x} + Be^{3x} + Ce^{4x}$ is the solution of a differential equation, then $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx}$ equals

(a) $2y$ (b) $-12y$ (c) $24y$ (d) $-24y$

Sol. **(c)** Differential equation whose solution is

$y = Ae^{2x} + Be^{3x} + Ce^{4x}$, is

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} = 24y$$

3. If $y = e^{kx}(A\cos mx + B\sin mx)$ is the solution of differential equation, where A and B are arbitrary constants, then corresponding differential equation is given by

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + (k^2 + m^2)y = 0$$

In particular if $k = 0$, then differential equation corresponding to $y = e^0(A\cos mx + B\sin mx)$

$$= A\cos mx + B\sin mx \text{ is } \frac{d^2y}{dx^2} + m^2y = 0$$

Example The differential equation whose solution is $y = e^x(a\cos 2x + b\sin 2x)$, is

(a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$

(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

(d) None of the above

Sol. **(c)** Differential equation, whose solution is

$y = e^x(a\cos 2x + b\sin 2x)$, is

$$\frac{d^2y}{dx^2} - 2\cdot 1\cdot\frac{dy}{dx} + (1^2 + 2^2)y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

4. If the solution of the differential equation is of the form,

(i) $y = A\cos(mx + B) = e^0A\cos(mx + B)$

(ii) $y = A\sin(mx + B) = e^0A\sin(mx + B)$

(iii) $y = A\sin(mx + B) + D\cos(mx + B)$

$= e^0\{A\sin(mx + B) + D\cos(mx + B)\}$, then the corresponding differential equation is $\frac{d^2y}{dx^2} + m^2y = 0$

Example The differential equation representing the family of curves $y = A\cos(x + B)$ is given by

(a) $\frac{d^2y}{dx^2} = k$ (b) $\frac{d^2y}{dx^2} - y = 0$

(c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$ (d) $\frac{d^2y}{dx^2} + y = 0$

Sol. ***(d)*** Clearly, the required differential equation is $\frac{d^2y}{dx^2} + 1^2 \cdot y = 0$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$$

5. If the solution of the differential equation is of the form,

(i) $y = (A + Bx)e^{kx}$

(ii) $y = (A + Bx + Cx^2)e^{kx}$

(iii) $y = (A + Bx + Cx^2)e^{-kx}$, then their corresponding differential equation, is

(a) $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0$

(b) $\frac{d^3y}{dx^3} - 3k\frac{d^2y}{dx^2} + 3k^2\frac{dy}{dx} - k^3y = 0$

(c) $\frac{d^3y}{dx^3} + 3k\frac{d^2y}{dx^2} + 3k^2\frac{dy}{dx} + k^3y = 0$ respectively

Example The differential equation whose solution is $y = (A + Bx)e^{3x}$, is

(a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ (b) $\frac{d^2y}{dx^2} + 9y = 0$

(c) $\frac{d^2y}{dx^2} - 9y = 0$ (d) None of these

Sol. ***(a)*** Clearly, the required differential equation is

$$\frac{d^2y}{dx^2} - 2 \cdot 3\frac{dy}{dx} + 9y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

6. Differential equation of family of all circles of radius r is

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

Example The differential equation of family of all circles of radius 3 is

(a) $(1 + y_1^2)^3 = 9y_2$

(b) $(1 + y_1^2)^3 = 9y_2^2$

(c) $(1 + y_1^2)^2 = 9y_2^3$

(d) None of the above

Sol. ***(b)*** Clearly, the required differential equation is

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = 3^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow \quad (1 + y_1^2)^3 = 9y_2^2$$

7. If $\frac{dy}{dx} = y\sin ax$, then the solution of differential equation is $ye^{(\cos ax)/a} = K$, where K is any constant.

or

If $\frac{dy}{dx} = y\cos ax$, then the solution of differential equation is $y = Ke^{(\sin ax)/a}$, where K is any constant.

Example If $y' = y\sin 3x$, then the solution of the differential equation is

(a) $y = Ke^{(\sin 3x)/3}$ (b) $y = Ke^{(\cos 3x)/3}$

(c) $ye^{(\sin 3x)/3} = K$ (d) $ye^{(\cos 3x)/3} = K$

Sol. ***(d)*** Here, $a = 3$

Hence, the solution of the given differential equation is

$$ye^{(\cos 3x)/3} = K.$$

8. If $\frac{dy}{dx} = y\tan ax$, then solution of differential equation is $y^a\cos ax = K$, where $K = c^a$, c is any constant.

Example The solution of differential equation $y' = y\tan 3x$ is

(a) $y^3\cos 3x = K$ (b) $y^3\sin 3x = K$

(c) $y^3\tan 3x = K$ (d) $y^3\cot 3x = K$

Sol. ***(a)*** Here, $a = 3$

Hence, the required solution is $y^3\cos 3x = K$.

9. If $\frac{dy}{dx} + \sin(ax \pm by) = \sin(ax \mp by)$, then solution of differential equation is

$$\frac{1}{b}\log|\operatorname{cosec} by - \cot by| \pm \frac{2}{a}\sin ax = K$$

where K is any constant.

Example The solution of

$$y' + \sin(2x - 3y) = \sin(2x + 3y) \text{ is}$$

(a) $\frac{1}{2}\log|\operatorname{cosec} 2y - \cot 2y| + \frac{2}{3}\sin 3x = K$

(b) $\frac{1}{3}\log|\operatorname{cosec} 3y - \cot 3y| - \sin 2x = K$

(c) $\frac{1}{2}\log|\operatorname{cosec} 2y - \cot 2y| - \frac{2}{3}\sin 3x = K$

(d) $\frac{1}{3}\log|\operatorname{cosec} 3y - \cot 3y| + \sin 2x = K$

Sol. ***(b)*** Here, $a = 2$ and $b = 3$

Hence, the required solution is

$$\frac{1}{3}\log|\operatorname{cosec} 3y - \cot 3y| - \sin 2x = K.$$

10. If $\frac{dy}{dx} - \sin(ax \pm by) = \sin(ax \mp by)$, then solution of differential equation is

$$\frac{1}{b}\log|\sec by + \tan by| + \frac{2}{a}\cos ax = K,$$

where K is any constant.

Example If $\frac{dy}{dx} - \sin(4x + 5y) = \sin(4x - 5y)$, then solution of differential equation is

(a) $\frac{1}{4}\log|\sec 4y + \tan 4y| + \frac{2}{5}\cos 5x = K$

(b) $\frac{1}{5}\log|\sec 5y + \tan 5y| - \frac{1}{2}\cos 4x = K$

(c) $\frac{1}{4}\log|\sec 4y + \tan 4y| - \frac{2}{5}\cos 5x = K$

(d) $\frac{1}{5}\log|\sec 5y + \tan 5y| + \frac{1}{2}\cos 4x = K$

Sol. ***(d)*** Here, $a = 4$ and $b = 5$

Hence, the required solution is

$\frac{1}{5}\log|\sec 5y + \tan 5y| + \frac{1}{2}\cos 4x = K$

11. If $\frac{dy}{dx} + \cos(ax \pm by) = \cos(ax \mp by)$, then solution of differential equation is

$\frac{1}{b}\log|\text{cosec } by - \cot by| \pm \frac{2}{a}\cos ax = K$, where K is any constant.

Example The solution of the differential equation $y' + \cos(x + 2y) = \cos(x - 2y)$ is

(a) $\frac{1}{2}\log|\text{cosec } 2y - \cot 2y| - 2\cos x = K$

(b) $\log|\text{cosec } x - \cot x| + \cos y = K$

(c) $\frac{1}{2}\log|\text{cosec } 2y - \cot 2y| + 2\cos x = K$

(d) $\log|\text{cosec } x - \cot x| - \cos y = K$

Sol. ***(c)*** Here, $a = 1$ and $b = 2$

Hence, the required solution is

$\frac{1}{2}\log|\text{cosec } 2y - \cot 2y| + 2\cos x = K$.

12. If $\frac{dy}{dx} - \cos(ax \pm by) = \cos(ax \mp by)$, then solution of the differential equation is

$\frac{1}{b}\log[\sec by + \tan by] - \frac{2}{a}\sin ax = K$,

where K is any constant.

Example The solution of $y' - \cos(2x - y) = \cos(2x + y)$ is

(a) $\frac{1}{2}\log|\sec 2y + \tan 2y| - 2\sin x = K$

(b) $\log|\sec y + \tan y| - \sin 2x = K$

(c) $\log|\sec 2y + \tan 2y| + 2\sin x = K$

(d) None of the above

Sol. ***(b)*** Here, $a = 2$ and $b = 1$

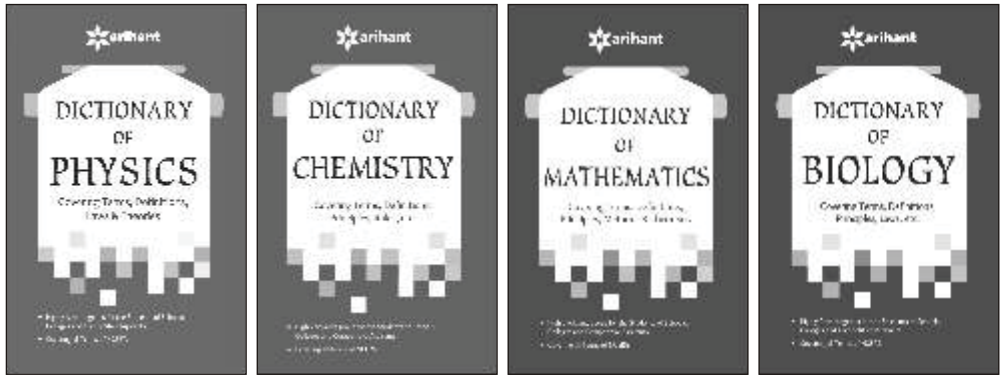
Hence, the required solution is

$\log|\sec y + \tan y| - \sin 2x = K$.

SCIENCE DICTIONARIES *in*

(Physics, Chemistry, Mathematics & Biology)

arihant



COVERING
Terms
Definitions
Concepts
Methods
Laws
Scientists & their works and experiments.

TARGET CBSE 2016

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Simulator Test Series to Zoom up Your Board Preparations

Board Exam Scale UP

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Section A

1 Mark Questions

1. Write the principal value of

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right).$$

2. Let $*$ be a binary operation on set of integers I, defined by $a * b = 3a + 4b - 2$, then find the value of $4 * 5$.

3. If $A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$, then find A^4.

4. Find the approximate change in the surface area of a cube of side x m caused by decreasing the side by 1%.

5. If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + e^{y-x} = 0$.

6. Find the distance of the plane $3x - 4y + 12z = 3$ from the origin.

[CBSE 2012]

Section B

4 Marks Questions

7. The cooperative store of a particular school has 10 dozen Physics books, 8 dozen Chemistry books and 5 dozen Mathematics books. Their selling prices are ₹ 65.70, ₹ 43.20 and ₹ 76.50, respectively. Find by matrix method the total amount received by the store from selling all these items.

8. If $A=\begin{bmatrix}1&2&2\\2&1&-2\\x&2&y\end{bmatrix}$ is a matrix satisfying $AA' = 9I$, then find the values of x and y.

9. If the function $f : R \to R$ is given by $f(x) = x^2 + 2$ and $g : R \to R$ is given by $g(x) = \frac{x}{x-1}$; $x \neq 1$, then find fog and gof and hence find fog (2) and gof (–3).

or

If $f : X \to Y$ is a function. Define a relation R on X given by $R = \{(a, b) : f(a) = f(b)\}$. Show that R is an equivalence relation on X.

10. The probabilities of two students A and B coming to school in time are 3/7 and 5/7, respectively. Assuming that the events, 'A is coming in time' and 'B is coming in time' are independent, find the probability of only one of them coming to the school in time. Write atleast one advantage of coming to the school in time.

[VBQ; CBSE 2013]

11. Using properties of determinants, show that

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0.$$

or

Using properties of determinants, show that

$$\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2.$$

12. Solve for x;

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}, -1 < x < 1$$

[CBSE 2011]

13. Find the equation of the plane through the point $\hat{i} + 4\hat{j} - 2\hat{k}$ and perpendicular to the line of intersection of planes

$\mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 10$ and $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 18$.

14. Evaluate $\int \frac{1}{3 + 2\sin x + \cos x}\, dx$.

or

Evaluate $\int \frac{x^2}{(x \sin x + \cos x)^2}\, dx$.

15. Form the differential equation representing family of curves given by $(x-a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

16. Evaluate $\int_0^{\pi} \frac{x}{(a^2 \cos^2 x + b^2 \sin^2 x)}\, dx$.

17. Differentiate $x^{\sin^{-1} x}$ with respect to $\sin^{-1} x$.

or

If $y = 3e^{2x} + 2e^{3x}$, then prove that

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

18. Show that $f(x) = |x - 2|$ is continuous but not differentiable at $x = 2$.

19. Solve the differential equation

$$x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x\,;\quad x \neq 0.$$

Section C

6 Marks Questions

20. Find the vector equation of the plane passing through the intersection of planes $\mathbf{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\mathbf{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and passing through the point (2, 1, 3).

or

Find the distance between the point (7, 2, 4) and the plane determined by the points A (2, 5, – 3), B (–2, –3, 5) and C (5, 3, –3).

21. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, show that $(1+|\mathbf{a}|^2)(1+|\mathbf{b}|^2) = (1 - \mathbf{a}\cdot\mathbf{b})^2 + |\mathbf{a} + \mathbf{b} + (\mathbf{a}\times\mathbf{b})|^2$.

22. Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is strictly decreasing or strictly increasing.

23. Show that the maximum area of triangle that can be inscribed in a given circle is an equilateral triangle.

24. Using method of integration find the area bounded by the curve $|x| + |y| = 1$.

25. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident involving a scooter driver, car driver and a truck driver are 0.01, 0.03 and 0.15, respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

[CBSE 2014, 12]

or

A card from a pack of 52 cards is lost, from the remaining cards of the pack two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart. [CBSE 2010]

26. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5.00 per kg to purchase food I and ₹ 7.00 per kg to produce food II. Determine the minimum cost to such a mixture. Formulate the above as a LPP and solve it. [CBSE 2012]

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Section A

1 Mark Questions

1. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, then find k so that $A^2 = 8A + kI$.

2. Write the principal value of

$$\left[\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)\right].$$

3. Let $* : R \times R \to R$ is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$.

4. Using derivative, find the approximate change in the volume V of a cube of side x m caused by increasing the side by 1%.

5. Write the derivative of sin x with respect to cos x.

6. Find the coordinates of a point, where the line $\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5}$ cuts YZ-plane.

Section B

4 Marks Questions

7. Express the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

8. Using properties of determinants, prove that $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$

or

Using properties of determinants, prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab.$

9. Evaluate $\int \frac{(3\sin\theta - 2)\cos\theta}{(5 - \cos^2\theta - 4\sin\theta)}\, d\theta.$

or

Evaluate $\int \frac{\sin x}{\sin 3x}\, dx.$

10. Prove that $\int_0^{\pi/4} \log(1 + \tan x)dx = \frac{\pi}{8}(\log 2).$ **[CBSE 2013, 11]**

11. *A* and *B* appear for an interview for two posts. The probability of *A*'s selection is (1/3) and that of *B*'s selection is (2/5). Find the probability that only one of them will be selected.

12. If the function $f : R \to R$ is given by $f(x) = x^2 + 3x + 1$ and $g : R \to R$ is given by $g(x) = 2x - 3$, then find (i) *fog* and (ii) *gof*.

or

Let N be the set of all natural numbers and let R be a relation on $N \times N$, defined by $(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$. Show that R is an equivalence relation.

13. Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y - z = 0$. [CBSE 2009, 11]

14. If $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x$, then find x.

15. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that

$$(1 + x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0.$$

[CBSE 2011, 13]

or

If $x = a\sin 2t(1 + \cos 2t)$ and $y = b\cos 2t(1 - \cos 2t)$, then show that

$$\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$$

16. Solve the differential equation $\text{cosec } x \log y \frac{dy}{dx} + x^2y^2 = 0$.

17. Solve the following differential equation

$$\frac{dy}{dx} + y\cot x = 4x\,\text{cosec } x,$$

given that $y = 0$, when $x = \frac{\pi}{2}$.

18. Show that the function defined as follows is continuous at $x = 2$ but not differentiable at $x = 2$

$$f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases}.$$

19. A store in a mall has three dozen shirts with 'SAVE ENVIRONMENT' printed, two dozen shirts 'SAVE TIGER' printed and five dozen shirts with 'GROW PLANTS' printed. The cost of each shirt is ₹ 595, ₹ 610 and ₹ 795 respectively. All these items were sold in a day. Find total collection of the store using matrix method.

Section C

6 Marks Questions

20. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then

(i) show that the angle θ between the vectors $\mathbf{b}$ and $\mathbf{c}$ is given by $\cos\theta = \frac{|\mathbf{a}|^2 - |\mathbf{b}|^2 - |\mathbf{c}|^2}{2|\mathbf{b}||\mathbf{c}|}$

(ii) Prove that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

21. Find the area of the region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$.

22. Find the intervals in which the function f is given by $f(x) = \sin x - \cos x,\ 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

23. Show that lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar. Also, find the equation of the plane containing these lines. [CBSE 2013]

or

Find the length and the foot of the perpendicular from the point $P(7, 14, 5)$ to the plane $2x + 4y - z = 2$. Also find the image of the point P in the plane.

24. Three critics review a book. For the three critics, the odds in favour of the book are (5 : 2), (4 : 3) and (3 : 4) respectively. Find the probability that the majority is in favour of the book.

or

Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of aces.

25. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.

26. Due to a state government campaign of energy saving for future generation, a man decided to ride his motorcycle at the speed of 50 km/h. He was to spend ₹ 2 per km on petrol. If he rides it at a faster speed of 80 km/h, the petrol cost increases to ₹ 3 per km. He has atmost ₹ 120 to spend on petrol and one hour's time. He wishes to find the maximum distance that he can travel. Express this problem as a linear programming problem. What are the values depict from man's action? [VBQ]

Hints of Board Exam Corner (February Issue)

1. We have,

$$(\sqrt{3}\,\hat{\mathbf{a}} - \hat{\mathbf{b}})^2 = 3\hat{\mathbf{a}}^2 + \hat{\mathbf{b}}^2 - 2\sqrt{3}\,\hat{\mathbf{a}}\hat{\mathbf{b}}$$

$$\Rightarrow \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \frac{\sqrt{3}}{2} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°$$

2. Unit vector in the direction of $\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$

$$= \frac{4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{26}}$$

Let $\mathbf{b}$ be a vector in the direction of $\mathbf{a}$, whose magnitude is 3, then

$$\mathbf{b} = |\mathbf{b}|\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{\sqrt{26}}(4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

3. We have, $2x - 3y + 5z + 4 = 0$

$$\Rightarrow \frac{x}{-2} + \frac{y}{\frac{4}{3}} + \frac{z}{\left(-\frac{4}{5}\right)} = 1$$

So, the intercepts are $-2, \frac{4}{3}$ and $-\frac{4}{5}$.

4. We have,

$$\frac{dy}{dx} = -\left(\frac{1+y^2}{y^2}\right) \Rightarrow \frac{dy}{dx}\left(\frac{1}{1+y^2} - 1\right) = 1$$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}$$

$\Rightarrow x + y = \tan^{-1} y$ is the required solution.

5. We have, $y^2 = a(b^2 - x^2)$

$$\Rightarrow \quad y\frac{dy}{dx} = -ax \qquad \ldots(i)$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \qquad \ldots(ii)$$

Using Eqs. (i) and (ii), we get

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

which is the required differential equation.

6. Since, $(AB' - BA')' = (AB')' - (BA')'$

$$= (BA' - AB') = -(AB' - BA')$$

Hence, $(AB' - BA')$ is a skew-symmetric matrix. **Hence proved.**

7. Given, $3A + 5B + 2C = O$

Now, let $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and find the value of a, b, c and d using given matrix equation.

$$\Rightarrow \qquad C = \begin{bmatrix} -24 & -10 \\ -28 & -38 \end{bmatrix}$$

or

Consider, $A' = A^{-1} \Rightarrow AA' = AA^{-1}$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\because AA^{-1} = I]$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{6}} \text{ and } z = \pm -\frac{1}{\sqrt{3}}$$

8. Let $4x + 5 = A\frac{d}{dx}(2x^2 + x - 3) + B$

Compare the coefficients of x and constant from both sides to find A and B.

Then, $I = \int \frac{(4x+5)}{\sqrt{2x^2 + x - 3}}dx$

$$= \int \frac{(4x+1)}{\sqrt{2x^2 + x - 3}}dx + 4\int \frac{dx}{\sqrt{2x^2 + x - 3}}$$

$$\Rightarrow \quad I = 2\sqrt{2x^2 + x - 3}$$

$$+ 2\sqrt{2}\log\left|\left(x + \frac{1}{4}\right) + \sqrt{\left(x + \frac{1}{4}\right)^2 - \left(\frac{5}{4}\right)^2}\right| + C$$

or

$$I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = \int \frac{\sqrt{1-\sqrt{x}} \times \sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}} \times \sqrt{1-\sqrt{x}}}dx$$

[multiplying numerator and denominator by $\sqrt{1-\sqrt{x}}$]

$$= \int \frac{(\sqrt{1-\sqrt{x}})^2}{\sqrt{1+\sqrt{x}} \times \sqrt{1-\sqrt{x}}}dx$$

$$= \int \frac{1-\sqrt{x}}{\sqrt{1-x}}dx = \int \frac{1}{\sqrt{1-x}}dx - \int \frac{\sqrt{x}}{\sqrt{1-x}}dx$$

$$I = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

9. (i) $\because P(A) = P(A \cap B) + P(A \cap \bar{B})$

$\therefore$ RHS $= P(A \cap B) + P(A \cap \bar{B})$

$= P(A) \cdot P(B) + P(A) \cdot P(\bar{B})$

$= P(A)[P(B) + P(\bar{B})] = P(A)[P(B) + 1 - P(B)]$

$[\because P(\bar{B}) = 1 - P(B)]$

$= P(A) =$ LHS **Hence proved.**

(ii) $\because P(A \cup B) = P(A \cap B) + P(A \cap \bar{B})$

$+ P(\bar{A} \cap B)$

$\therefore$ RHS $= P(A) \cdot P(B) + P(A) \cdot P(\bar{B})$

$+ P(\bar{A}) \cdot P(B)$

$= P(A) \cdot P(B) + P(A) - P(A) \cdot P(B)$

$+ P(B) - P(A) \cdot P(B)$

$= P(A) + P(B) - P(A) \cdot P(B)$

$= P(A) + P(B) - P(A \cap B)$

$= P(A \cup B) =$ LHS **Hence proved.**

10. First a unit vector which is perpendicular to both $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

Let $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and $\mathbf{a} - \mathbf{b} = \mathbf{d}$, then

$$\mathbf{n} = \frac{\mathbf{c} \times \mathbf{d}}{|\mathbf{c} \times \mathbf{d}|} \quad \therefore \mathbf{n} = \frac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{6}}$$

$$\therefore \text{ Required vector} = \frac{-5}{\sqrt{6}}\hat{\mathbf{i}} + \frac{10}{\sqrt{6}}\hat{\mathbf{j}} - \frac{5}{\sqrt{6}}\hat{\mathbf{k}}$$

11. Take logarithm on both sides, i.e. write given equations as $\log x = \frac{1}{2}\log a^{\sin^{-1} t}$

and $\log y = \frac{1}{2}\log a^{\cos^{-1} t}$

Now, consider $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

12. Shortest distance

$$= \text{SD} = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$$

$$= \left|\frac{-9}{\sqrt{1+25+169}}\right| = \frac{9}{\sqrt{195}} \neq 0$$

Hence, the given lines do not intersect.

13. We have,

$$\tan^{-1}\frac{x-1}{x-2} = \tan^{-1} 1 - \tan^{-1}\frac{x+1}{x+2}$$

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}\left(\frac{1}{2x+3}\right) \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

or Let $\sin^{-1}\frac{5}{13} = x$ and $\cos^{-1}\frac{3}{5} = y$

$$\Rightarrow \qquad \cos x = \frac{12}{13} \text{ and } \sin y = \frac{4}{5}$$

Now, find $\tan x$ and $\tan y$, and then apply the formula $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$

14. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \frac{-1}{2}(a+b+c)$

$[(a-b)^2 + (b-c)^2 + (c-a)^2]$

Now, $\Delta = 0$

$$\Rightarrow \frac{-1}{2}(a+b+c)[(a-b)^2 + (b-c)^2$$

$$+ (c-a)^2] = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$[\because a + b + c \neq 0$, given$]$

$\Rightarrow \quad a = b = c$ **Hence proved.**

15. Put $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$

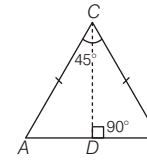
When $x = 0$, then $\theta = 0$ and when $x = 1$, then $\theta = \frac{\pi}{4}$. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx$.

Then, $I = \int_0^{\pi/4} \log(1 + \tan\theta)d\theta = \frac{\pi}{8}\log 2$

16. Let two men start from the point C with velocity v each at the same time.

Also, $\angle BCA = 45°$

Since, A and B are moving with same velocity v, so they will cover same distance in same time.



Therefore, ΔABC is an isosceles triangle with $AC = BC$.

Now, draw $CD \perp AB$

Let at any instant t, the distance between them is AB.

Let $AC = BC = x$ and $AB = y$

$$\therefore \angle ACD = \frac{\pi}{8}$$

$$\therefore \sin\frac{\pi}{8} = \frac{AD}{AC} \Rightarrow \sin\frac{\pi}{8} = \frac{y/2}{x} \quad [\because AD = y/2]$$

$$\Rightarrow \quad y = 2x \cdot \sin\frac{\pi}{8}$$

Now, differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = 2 \cdot \sin\frac{\pi}{8} \cdot \frac{dx}{dt} = \sqrt{2-\sqrt{2}}\ v \text{ unit/s}$$

$$[\because v = \frac{dx}{dt} \text{ and } \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}]$$

which is the rate at which A and B are being separated.

17. LHS

$$= \frac{1}{2}\left[\left\{\sqrt{(a^2 - x^2)} + \frac{x}{2\sqrt{a^2 - x^2}}(-2x)\right\} + a^2\left\{\frac{a}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}\right\}\right]$$

$$= \frac{1}{2}\left[\left\{\frac{(a^2 - x^2) - x^2}{\sqrt{a^2 - x^2}}\right\} + \frac{a^2}{\sqrt{a^2 - x^2}}\right]$$

$$= \sqrt{a^2 - x^2} = \text{RHS} \qquad \textbf{Hence proved.}$$

or

Given, $y=\{\log(x+\sqrt{x^2+1})\}^2$ ...(i)

On differentiating w.r.t. 'x', we get

$$(\sqrt{x^2+1})y_1=2\log(x+\sqrt{x^2+1})$$

On squaring both sides, we get

$$(x^2+1)y_1^2=4\{\log(x+\sqrt{x^2+1})\}^2$$

$\Rightarrow \quad (x^2+1)y_1^2=4y$ [from Eq. (i)]

On differentiating w.r.t. 'x', we get

$$(x^2+1)\times 2y_1y_2+2xy_1^2=4y_1$$

$\Rightarrow (x^2+1)y_2+xy_1=2$ **Hence proved.**

18. $I=\int\frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

$$=\int\frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{1-2\left(2\cos^2\frac{3x}{2}-1\right)}dx$$

$$=-\int\frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}\cdot\cos\frac{3x}{2}}{4\cos^3\frac{3x}{2}-3\cos\frac{3x}{2}}dx$$

$$\left[\text{multiply and divide by }\cos\frac{3x}{2}\right]$$

$$=-\int\frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}\cdot\cos\frac{3x}{2}}{\cos 3\cdot\frac{3x}{2}}dx$$

$$=-\frac{1}{2}\sin 2x-\sin x+C$$

19. According to the given condition, we have

$$4x+3y+2z=37000$$

$$5x+3y+4z=47000$$

and $\quad x+y+z=12000$

The system of equations can be written in matrix form as

$$\begin{bmatrix}4&3&2\\5&3&4\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}37000\\47000\\12000\end{bmatrix}$$

i.e. $AX=B$, where

$$A=\begin{bmatrix}4&3&2\\5&3&4\\1&1&1\end{bmatrix},\ B=\begin{bmatrix}37000\\47000\\12000\end{bmatrix}\text{ and }X=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Now, $|A|\neq 0$ and $\text{Adj}(A)=\begin{bmatrix}-1&-1&6\\-1&2&-6\\2&-1&-3\end{bmatrix}$

$\because\ X=A^{-1}\cdot B$

$$\therefore\ X=\frac{-1}{3}\begin{bmatrix}-1&-1&6\\-1&2&-6\\2&-1&-3\end{bmatrix}\begin{bmatrix}37000\\47000\\12000\end{bmatrix}$$

$\Rightarrow x=4000,\ y=5000,\ z=3000$

The values described in question are resourcefulness, competence and determination.

20. Clearly, $A=\{0,1,2,3,4,\ldots,10,11,12\}$

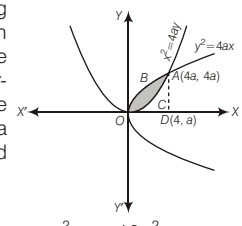
Here, R satisfies the following properties

(i) **Reflexivity**
Let a be an arbitrary element of A. Then, $a-a=0$, which is a multiple of 4. So, R is reflexive.

(ii) **Symmetry** Let $a\,Rb$.
Then, $aRb\Rightarrow|a-b|$ is a multiple of 4
$\Rightarrow|-(b-a)|$ is a multiple of 4
$\Rightarrow|b-a|$ is a multiple of 4
So, R is symmetric.

(iii) **Transitivity**
Let aRb and bRc. Then, $aRb,\ bRc$
$\Rightarrow|a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4
Let $|a-b|=4k_1$ and $|b-c|=4k_2$.
Then, $|a-c|=|(a-b)+(b-c)|$
$=|\pm 4k_1\pm 4k_2|$
$=|4(\pm k_1\pm k_2)|=4|\pm k_1\pm k_2|$, which is a multiple of 4.
So, R is transitive.
Hence, R is an equivalence relation.
Now, $[1]=\{x\in A:|x-1|$ is a multiple of $4\}=\{1,5,9\}$
Hence, the required set is $\{1,5,9\}$.

21. On solving both the given equations, we get the intersection of the two parabola as $O(0,0)$ and $A(4a,4a)$.



Required area

$$=\int_0^{4a}2\sqrt{ax}\ dx-\int_0^{4a}\frac{x^2}{4a}dx=\frac{16a^2}{3}\text{ sq units}$$

or

Let $f(x)=(3x^2+2x+1);\ a=0,\ b=1$

and $nh=(1-0)=1$

Then, $\int_0^1(3x^2+2x+1)dx$

$$=\lim_{h\to 0}h[f(0)+f(0+h)+f(0+2h)+\ldots+f\{0+(n-1)h\}]$$

$$=\lim_{h\to 0}h[n+3h^2\{1^2+2^2+3^2+\ldots+(n-1)^2\}+2h\{1+2+3+\ldots+(n-1)\}]$$

$$=\lim_{h\to 0}h\left[n+3h^2\cdot\frac{(n-1)n(2n-1)}{6}+2h\cdot\frac{n(n-1)}{2}\right]=3$$

22. The given differential equation is linear,

where $\text{IF}=e^{\int P\,dy}=e^{\int\frac{1}{(1+y^2)}dy}=e^{\tan^{-1}y}$

Now, the solution of the given differential equation is given by

$x\times\text{IF}=\int\{Q\times\text{IF}\}dy+C$, where C is an arbitrary constant.

$\Rightarrow x=(\tan^{-1}y-1)+e^{-\tan^{-1}y}$, which is the required solution.

or We have,

$$\frac{dy}{dx}=\frac{-2xy}{(x^2+1)}\Rightarrow\int\frac{dy}{y}=\int\frac{-2x}{(x^2+1)}dx$$

[integrating both sides]

$\Rightarrow \quad y(x^2+1)=C$ ...(i)

Now, it is given that the curve passes through $(1,2)$. So, putting $x=1$ and $y=2$ in Eq. (i), we get $C=4$

$\therefore\ y(x^2+1)=4$ is the required equation of the curve.

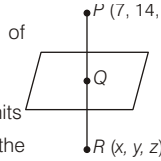
23. The equation of the line passing through the point $P(7,14,5)$ and perpendicular to the given plane is given by

$$\frac{x-7}{2}=\frac{y-14}{4}=\frac{z-5}{-1}=\lambda\quad\text{(say)}$$

A general point on this line is $(2\lambda+7,4\lambda+14,-\lambda+5)$. For some value of λ, let the point $Q(2\lambda+7,4\lambda+14,-\lambda+5)$ lie on the plane.

$\Rightarrow \quad \lambda=-3$

$\therefore$ The coordinates of Q is $(1,2,8)$.



Now, $PQ=\sqrt{189}=3\sqrt{21}$ units

Let $R(x,y,z)$ be the image of P in the plane.

Then, Q is the mid-point of PR.

$$\therefore\frac{7+x}{2}=1,\frac{14+y}{2}=2\text{ and }\frac{5+z}{2}=8$$

Hence, the image of $P(7,14,5)$ in the given plane is $R(-5,-10,11)$.

24. Let A, B, C be the given students and let E_1, E_2 and E_3 be the events that the problem is solved by A, B, C, respectively.

Then, $\quad P(E_1)=\frac{1}{2};\ P(E_2)=\frac{1}{3};\ P(E_3)=\frac{1}{4};$

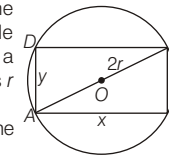
$$P(\bar{E}_1)=\left(1-\frac{1}{2}\right)=\frac{1}{2};\ P(\bar{E}_2)=\left(1-\frac{1}{3}\right)=\frac{2}{3}$$

and $P(\bar{E}_3)=\left(1-\frac{1}{4}\right)=\frac{3}{4}$

Required probability

$$=\left(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\right)+\left(\frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}\right)+\left(\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}\right)=\frac{11}{24}$$

25. Let x and y be the sides of the rectangle $ABCD$ inscribed in a given circle of radius r with centre O.



Clearly, $AC=2r$ is the diameter of the circle.

Then, $x^2+y^2=(2r)^2$

[$\because\angle B$ is a right angled]

$\Rightarrow \quad y^2=4r^2-x^2$ ...(i)

Let A be the area of rectangle, then

$$A=xy\ \Rightarrow A^2=x^2y^2$$

$\Rightarrow \quad z=x^2y^2$, where $z=A^2$

$\Rightarrow \quad z=x^2[4r^2-x^2]=4r^2x^2-x^4$

[using Eq. (i)]

Now, z is maximum when the rectangle is a square of side $\sqrt{2}\,r$

$\Rightarrow A^2$ is maximum when the rectangle is a square of side $\sqrt{2}\,r$

$\Rightarrow A$ is maximum when the rectangle is a square of side $\sqrt{2}\,r$

26. Let x units of A and y units of B be produced in order to have a maximum profit. Then, clearly $x\geq 0$ and $y\geq 0$

x units of A and y units of B will take $(x+y)$ minutes on M_1. $\quad\therefore x+y\leq 300$

x units of A and y units of B will take $(2x+y)$ minutes on M_2. $\quad\therefore 2x+y\leq 360$

Let Z be the profit function.

Then, $\quad Z=5x+3y$

We have to maximise $Z=5x+3y$,

Subject to the constraints

$x\geq 0,\ y\geq 0,\ x+y\leq 300$ and $2x+y\leq 360$.

Z is maximum when $x=60$ and $y=240$.

Hints of Board Exam Corner (February Issue)

1. Projection of $\mathbf{a}$ over $\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

$$= \frac{8}{\sqrt{(2)^2+(6)^2+(3)^2}}$$

$[\because \mathbf{a}\cdot\mathbf{b} = 8 \text{ and } \mathbf{b} = 2\hat{\mathbf{i}}+6\hat{\mathbf{j}}+3\hat{\mathbf{k}}]$

$$= \frac{8}{\sqrt{49}} = \frac{8}{7}$$

2. At $k = 0, |k\mathbf{a}| = 0$ and at $k = 2, |k\mathbf{a}| = 6$

$\Rightarrow |k\mathbf{a}|$ belongs to the interval $[0, 6]$

3. We can write the given differential equation as $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{\sin x}{x}$

$\therefore$ IF $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = \dfrac{1}{x}$

4. Total number of possible matrices of order 3×3 with each entry 2 or 0 is 2^9 is 512.

5. We have, $y = A\cos(x+B)$

$\Rightarrow \dfrac{d^2y}{dx^2} = -A\cos(x+B) = -y$

$\Rightarrow \dfrac{d^2y}{dx^2} + y = 0$, which is the required differential equation.

6. The direction cosines are

$\cos\dfrac{\pi}{2}, \cos\dfrac{3\pi}{4}, \cos\dfrac{\pi}{4}$, i.e. $\pm\left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

7. $A^2 - 5A - 14I = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow A^2 - 5A - 14I = O \Rightarrow A^3 - 5A^2 - 14A = O$

$\Rightarrow \quad A^3 = 5A^2 + 14A \qquad [\because AI = A]$

$$= \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$$

or

To find A^{-1}, consider $A = IA$ and then apply elementary row operations to convert above equation as $I = BA$.

So, the inverse of A is

$$B = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}.$$

8. We have, $\int \dfrac{x^3+x+1}{x^2-1}dx$

$$= \int x\,dx + \int \frac{2x}{x^2-1}dx + \int \frac{1}{x^2-1}dx$$

$$= \frac{x^2}{2} + \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C$$

or

Consider $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{1+x^2}$

and find A, B and C to get

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

Integrating both sides, we get

$$\int \frac{2}{(1-x)(1+x^2)}dx$$

$$= -\log(1-x) + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + C$$

9. Clearly, $n(S) = 36$,

$n(E) = 3$, $n(F) = 10$ and $n(G) = 7$

Here, $(E \cap F) = \phi$ and $(E \cap G) = \phi$

Also, $(F \cap G) = \{(4, 6), (6, 4), (5, 5)\}$

$\therefore \quad n(F \cap G) = 3$ and $(E \cap F \cap G) = \phi$

$$P(E) = \frac{1}{12};\ P(F) = \frac{5}{18};\ P(G) = \frac{7}{36}$$

$$P(F \cap G) = \frac{1}{12} \text{ and } P(F)\cdot P(G) = \frac{35}{648}$$

Here, we see that $P(F \cap G) \neq P(F)\cdot P(G)$

Hence, there is no pair which is independent.

10. Use the formula,

Area of triangle $= \dfrac{1}{2}|\mathbf{AB} \times \mathbf{AC}|$

where, $\mathbf{AB}$ and $\mathbf{AC}$ are the adjacent sides of the triangle.

$\therefore$ Area of triangle $= \dfrac{1}{2}\sqrt{61}$ sq units

or

It is given that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are non-coplanar vectors.

$\therefore [\mathbf{A}\,\mathbf{B}\,\mathbf{C}] \neq 0$

$$\Rightarrow \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \quad \text{...(i)}$$

Now, $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\Rightarrow \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\Rightarrow (1+abc)[\mathbf{A}\,\mathbf{B}\,\mathbf{C}] = 0$ [using Eq. (i)]

$\Rightarrow \quad abc = -1$ **Hence proved.**

11. We have, LHS $= \sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$

$= \sin[\cot^{-1}\{\cos A\}]$, where $\tan^{-1}x = A$ or $x = \tan A$

$$= \sin\left[\cot^{-1}\left\{\frac{1}{\sec A}\right\}\right]$$

$$= \sin\left[\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right] \qquad [\because \tan A = x]$$

$= \sin B$, where $\cot^{-1}\dfrac{1}{\sqrt{1+x^2}} = B$

or $\quad \cot B = \dfrac{1}{\sqrt{1+x^2}} = \dfrac{1}{\sqrt{1+\frac{1}{1+x^2}}}$

$$\left[\because \sin B = \frac{1}{\operatorname{cosec} B};\ \cot B = \frac{1}{\sqrt{1+x^2}}\right.$$

$$\left.\text{and } \operatorname{cosec}^2 B = 1 + \cot^2 B\right]$$

$$= \sqrt{\frac{x^2+1}{x^2+2}} = \text{RHS}$$ **Hence proved.**

or

Let $\sin^{-1}\dfrac{3}{5} = A \quad \Rightarrow \quad \sin A = \dfrac{3}{5}$ and

$\cot^{-1}\dfrac{3}{2} = B \Rightarrow \cot B = \dfrac{3}{2} \quad \therefore \cos A = \dfrac{4}{5}$

and $\sin B = \dfrac{1}{\operatorname{cosec} B} = \dfrac{1}{\sqrt{1+\cot^2 B}} = \dfrac{2}{\sqrt{13}}$

Also, $\cos B = \sqrt{1-\sin^2 B} = \dfrac{3}{\sqrt{13}}$

Now, $\cos(A+B) = \cos A\cos B - \sin A\sin B = \dfrac{6}{5\sqrt{13}}$ **Hence proved.**

12. Clearly, $BA = 6I$; $\therefore B^{-1} = \dfrac{1}{6}A$

Also,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Hence, $x = 2$, $y = -1$ and $z = 4$

13. Given, $x = a(\cos t + t\sin t)$

$\Rightarrow \dfrac{d^2x}{dt^2} = a(\cos t - t\sin t)$

and $y = a(\sin t - t\cos t)$

$\Rightarrow \dfrac{d^2y}{dt^2} = a(\sin t + t\cos t)$

Now, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{at\sin t}{at\cos t} = \tan t$

Hence, $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3 t}{at}$

14. Put $x = a\tan^2\theta$, then

$dx = 2a\tan\theta\sec^2\theta\,d\theta$

and $\quad a + x = a + a\tan^2\theta = a(1+\tan^2\theta)$

$= a\sec^2\theta$

When $x = a$, then $a\tan^2\theta = a \Rightarrow \tan^2\theta = 1$

$\Rightarrow \quad \tan\theta = 1 \Rightarrow \quad \theta = \dfrac{\pi}{4}$

Given integral

$$= a\int_0^{\pi/4} \underset{\text{I}}{\theta}\,(2\tan\theta\underset{\text{II}}{\sec^2\theta})\,d\theta = \frac{a}{2}(\pi - 2)$$

Hence proved.

15. On comparing the given equations with $\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$, we get

$$\mathbf{a}_1 = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}, \mathbf{b}_1 = -\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$\mathbf{a}_2 = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{b}_2 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

Now, the shortest distance between the two given lines given by

$$\text{SD} = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$$

16. We have, $y = \log[x + \sqrt{x^2+1}]$ ...(i)

Differentiating both sides of Eq. (i) w.r.t. x,

we get $(x^2+1)\left(\dfrac{dy}{dx}\right)^2 = 1$ ...(ii)

Again, differentiating both sides of Eq. (ii) w.r.t. x, we get

$$(x^2+1)2\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + 2x\cdot\left(\frac{dy}{dx}\right)^2 = 0$$

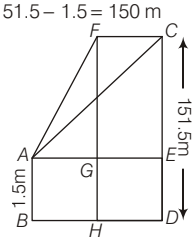
$$\Rightarrow \quad (x^2+1)\cdot\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

$\left[\because 2\dfrac{dy}{dx} \neq 0\right]$ **Hence proved.**

17. Let $DB = x$ m $= EA$ and $AC = y = 250$ m

$\therefore \frac{dx}{dt} = 10$ m/s

From the figure, we see that $EC = 151.5 - 1.5 = 150$ m



and $AE = x$. Also, $AC = 250$ m

In right angled ΔCEA, $AE^2 + EC^2 = AC^2$

$\Rightarrow \quad x^2 + (150)^2 = y^2 \qquad \ldots(i)$

$\Rightarrow \quad x = 200$

On differentiating Eq. (i) w.r.t. t, we get

$$\frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt} = \frac{200}{250} \cdot 10 = 8 \text{ m/s}$$

$$\left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

So, the required rate at which the string is being let out is 8 m/s.

18. **Hint** Apply

$$\frac{px+q}{(x-a)(x+b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x+b)} + \frac{C}{(x-c)},$$

then get the values of A, B and C and use

$$\int \frac{1}{x} dx = \log|x| + C.$$

$$I = \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C$$

19. According to the question, we have

$2x + 3y + 4z = 4600$,

$3x + 2y + 3z = 4100$; $x + y + z = 1500$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4600 \\ 4100 \\ 1500 \end{bmatrix}$$

Here, $|A| \neq 0$ and $\text{Adj}|A| = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 6 \\ 1 & 1 & -5 \end{bmatrix}$

$\therefore$ By using $X = A^{-1}B$, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \\ 600 \end{bmatrix}$

$\therefore$ A part from these three values, punctuality should be considered for the award.

20. (i) **Reflexivity** Let $(a, b) \in A \times A$. Then,

$(a, b) \in A \times A \Rightarrow a, b \in A \Rightarrow a + b = b + a$

$\Rightarrow \quad (a, b) R (a, b)$

$\therefore$ R is reflexive.

(ii) **Symmetry** Let $(a, b) R (c, d)$.

Then, $(a, b) R (c, d) \Rightarrow a + d = b + c$

$\Rightarrow c + b = d + a$

$\therefore$ R is symmetric.

(iii) **Transitivity** Let $(a, b) R (c, d)$ and $(c, d) R (e, f)$. Then,

$(a, b) R (c, d)$ and $(c, d) R (e, f)$

$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$

$\Rightarrow \quad a + d + c + f = b + c + d + e$

$\Rightarrow \quad a + f = b + e$

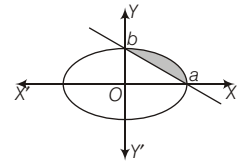
$\therefore R$ is transitive.

Hence, R is an equivalence relation

$[(2, 5)] = \{(a, b) : (2, 5) R (a, b)\}$

$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

21.



Required area

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b(a-x)}{a} dx$$

$$= \left(\frac{\pi ab}{4} - \frac{ab}{2}\right) \text{ sq units}$$

or

Let $f(x) = e^x$ and $nh = \{1 - (-1)\} = 2$. Then,

$$\int_{-1}^{1} e^x dx = \lim_{h \to 0} h[f(-1) + f(-1+h) + \ldots + f\{-1 + (n-1)h\}]$$

$$= \lim_{h \to 0} h[e^{-1} + e^{(-1+h)} + e^{(-1+2h)} + \ldots + e^{\{(-1+(n-1)h\}}]$$

$[\because f(-1) = e^{-1}, f(-1+h) = e^{(-1+h)}$, etc]

$$= \lim_{h \to 0} e^{-1} h[1 + e^h + e^{2h} + \ldots + e^{(n-1)h}]$$

$$= \frac{1}{e} \cdot \lim_{h \to 0} \frac{h(e^2 - 1)}{(e^h - 1)} \qquad [\because nh = 2]$$

$$= \frac{1}{e} \cdot \lim_{h \to 0} \frac{e^2 - 1}{\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\right)}$$

$$= \frac{1}{e}(e^2 - 1) = \left(e - \frac{1}{e}\right)$$

22. The equations of the given lines are

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} = \lambda \text{ (say)} \quad \ldots(i)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} = \mu \text{ (say)} \quad \ldots(ii)$$

Any point on the line (i) is $P(3\lambda + 8, -16\lambda - 9, 7\lambda + 10)$.

Any point on the line (ii) is $Q(3\mu + 15, 8\mu + 29, -5\mu + 5)$.

The direction ratios of PQ are

$(3\mu - 3\lambda + 7, 8\mu + 16\lambda + 38, -5\mu - 7\lambda - 5)$

Now, PQ will be the shortest distance between line (i) and line (ii) only when PQ is perpendicular to each one of line (i) and line (ii). Using this, we get $\lambda = -1, \mu = -2$

Now, $\lambda = -1$ gives $P(5, 7, 3)$ and $\mu = -2$ gives $Q(9, 13, 15)$.

$\therefore$ SD $= |PQ| = 14$ units

and the equation of the line of shortest distance is $\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

23. Here, $P = \cot y$ and $Q = (2y + y^2 \cot y)$

$\therefore \quad \text{IF} = e^{\int P dy} = e^{\int \cot y dy} = e^{\log \sin y} = \sin y$

Now, the solution is given by

$x \times \text{IF} = \int (Q \times \text{IF}) dy + C$,

$\Rightarrow \quad x = y^2 + C \text{ cosec } y \qquad \ldots(i)$

Putting $y = \frac{\pi}{2}$ and $x = 0$ in Eq. (i), we get

$$C = \frac{-\pi^2}{4}$$

Hence, the required solution is

$$x = y^2 - \frac{\pi^2}{4} \text{cosec } y$$

or

We have, $(1 + e^{2x}) dy + e^x (1 + y^2) dx = 0$

$$\Rightarrow \quad \int \frac{1}{(1+y^2)} dy + \int \frac{e^x}{(1+e^{2x})} dx = C$$

[integrating both sides]

$\Rightarrow \quad \tan^{-1} y + \tan^{-1} e^x = C \quad \ldots(i) \quad [\because t = e^x]$

Putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$C = (\tan^{-1} 1 + \tan^{-1} 1) = \left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{2}$$

$\therefore \tan^{-1} y + \tan^{-1} e^x = \pi/2$ is the required solution.

24. Let A, B, C denote the events that the book be favoured by the first, second and third critic, respectively.

$\therefore$ Required probability

$= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C)$

$+ P(\bar{A} \cap B \cap C) + P(A \cap B \cap C) = \frac{209}{343}.$

25. Let the length of one piece of a wire be x cm and the other piece of wire be y cm, so that

$x + y = 36$ cm $\qquad \ldots(i)$

Let S denote the sum of the areas of a square and an equilateral triangle, then

$$S = \frac{(36-y)^2}{16} + \frac{\sqrt{3}y^2}{36}$$

Differentiating both sides w.r.t. y, we get

$$\frac{dS}{dy} = \frac{2}{16}(36 - y)(-1) + \frac{\sqrt{3}}{36}(2y)$$

For maximum or minimum, we have

$$\frac{dS}{dy} = 0 \Rightarrow y = \frac{324}{9 + 4\sqrt{3}}$$

Now, $x = 36 - y \Rightarrow x = \frac{144\sqrt{3}}{9 + 4\sqrt{3}}$

$$\frac{d^2S}{dy^2} = \frac{1}{8} + \frac{\sqrt{3}}{18} > 0 \text{ for } y = \frac{324}{9 + 4\sqrt{3}}$$

Now, S is minimum when $y = \frac{324}{9 + 4\sqrt{3}}$

Hence, the length of one piece of a wire

$$= x = \frac{144\sqrt{3}}{9 + 4\sqrt{3}} \text{ cm}$$

and the length of the other piece of a wire

$$= y = \frac{324}{9 + 4\sqrt{3}} \text{ cm}$$

26. Let x kg of A and y kg of B be produced. Then,

$x \geq 0$, $y \geq 0$, $x + 2y \geq 80$ and $3x + y \geq 75$.

Also, cost function is given by $Z = 4x + 6y$.

Thus, we have to minimise $Z = 4x + 6y$, subject to the constraints.

$x \geq 0$, $y \geq 0$, $x + 2y \geq 80$ and $3x + y \geq 75$

Hence, for a minimum cost, 14 kg of A and 33 kg of B must be taken.

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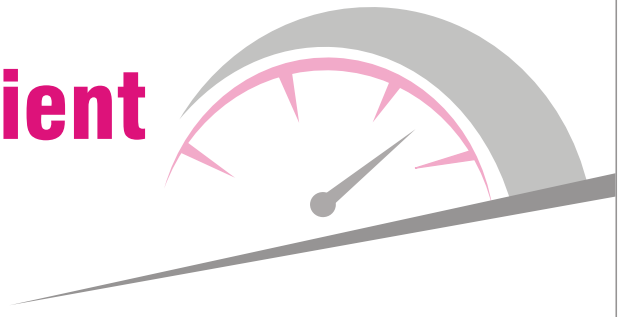
Knowledge Coefficient

Quizzer (No.16)



1. If α, β, γ and δ are four complex numbers such that γ/δ is real and $\alpha\delta - \beta\gamma \neq 0$, then $z = \dfrac{\alpha + \beta t}{\gamma + \delta t}$, $t \in R$ represents

(a) a circle (b) a parabola
(c) an ellipse (d) a straight line

2. A class contains 4 boys and g girls. Every sunday 5 students, including atleast 3 boys go for a picnic to Appu Ghar, a different group being sent every week. During, the picnic, the class teacher gives each girl in the group a doll. If the total number of dolls distributed was 85, then the value of g is

(a) 15 (b) 12 (c) 8 (d) 5

3. If A denotes the area bounded by $f(x) = \left|\dfrac{\sin x + \cos x}{x}\right|$, X-axis, $x = \pi$ and $x = 3\pi$, then

(a) $1<A<2$ (b) $0<A<2$ (c) $2<A<3$ (d) None of these

4. A circle with centre at the origin and radius equal to a, meets the X-axis at A and B. $P(\alpha)$ and $Q(\beta)$ are two points on this circle such that $\alpha - \beta = 2\gamma$, where γ is a constant. The locus of the point of intersection of AP and BQ is

(a) $x^2 - y^2 - 2ay\tan y = a^2$ (b) $x^2 + y^2 - 2ay\tan y = a^2$
(c) $x^2 + y^2 + 2ay\tan y = a^2$ (d) $x^2 - y^2 + 2ay\tan y = a^2$

5. If $y = \tan^{-1}\dfrac{1}{1 + x + x^2} + \tan^{-1}\dfrac{1}{x^2 + 3x + 3} + \tan^{-1}\dfrac{1}{x^2 + 5x + 7} + \ldots$ upto n terms, then $y'(0)$ is equal to

(a) $-\dfrac{1}{1 + n^2}$ (b) $-\dfrac{n^2}{1 + n^2}$
(c) $\dfrac{n}{1 + n^2}$ (d) None of these

6. If $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are three mutually perpendicular vectors of the same magnitude and if a vector $\mathbf{x}$ satisfies the equation $\mathbf{p} \times [(\mathbf{x} - \mathbf{q}) \times \mathbf{p}] + \mathbf{q} \times [(\mathbf{x} - \mathbf{r}) \times \mathbf{q}] + \mathbf{r} \times [(\mathbf{x} - \mathbf{p}) \times \mathbf{r}] = \mathbf{0}$, then vector $\mathbf{x}$ is

(a) $(1/2)(\mathbf{p} + \mathbf{q} - 2\mathbf{r})$ (b) $(1/2)(\mathbf{p} + \mathbf{q} + \mathbf{r})$
(c) $(1/3)(\mathbf{p} + \mathbf{q} + \mathbf{r})$ (d) $(1/3)(2\mathbf{p} + \mathbf{q} - \mathbf{r})$

7. A letter is known to have come from either TATANAGAR or CALCUTTA. On an envelope, just two consecutive letters TA are visible. The probability that the letter has come from CALCUTTA is

(a) 4/11 (b) 1/3 (c) 5/12 (d) 1/7

8. $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$, $y_1 < 0, y_2 < 0$ are the end points of the latusrectum of the ellipse $x^2 + 4y^2 = 4$. The equation of the parabola with latusrectum AB is

(a) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
(b) $x^2 - 2\sqrt{3}y = -3 + \sqrt{3}$
(c) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
(d) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

9. Let $f(x) = \dfrac{x}{(1 + x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = fof\ldots of(x)$ (n times). Then, $\int x^{n-2}\, g(x)\, dx$ equals to

(a) $\dfrac{1}{n(n-1)}(1 + nx^n)^{1 - \frac{1}{n}} + K$ (b) $\dfrac{1}{(n-1)}(1 + nx^n)^{1 - \frac{1}{n}} + K$

(c) $\dfrac{1}{n(n-1)}(1 + nx^n)^{1 + \frac{1}{n}} + K$ (d) $\dfrac{1}{(n-1)}(1 + nx^n)^{1 + \frac{1}{n}} + K$

10. Let $\alpha = \dfrac{\pi}{5}$ and $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $B = A + A^2 + A^3 + A^4$ is

(a) singular (b) non-singular
(c) skew-symmetric (d) $|B| = 1$

KNOWLEDGE Coefficient

Quizzer (No. 16)

Q. No.	a.	b.	c.	d.	Q. No.	a.	b.	c.	d.
1.	○	○	○	○	6.	○	○	○	○
2.	○	○	○	○	7.	○	○	○	○
3.	○	○	○	○	8.	○	○	○	○
4.	○	○	○	○	9.	○	○	○	○
5.	○	○	○	○	10.	○	○	○	○

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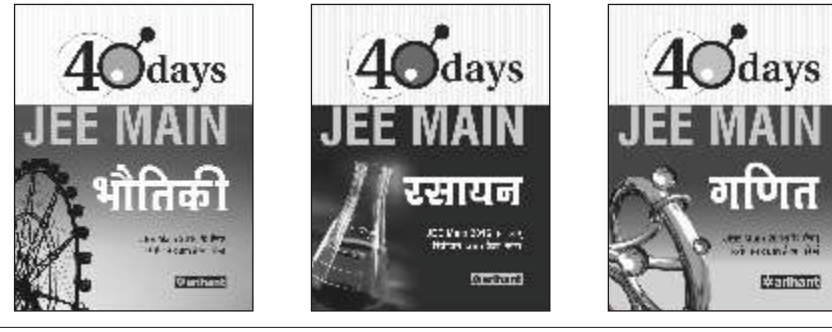

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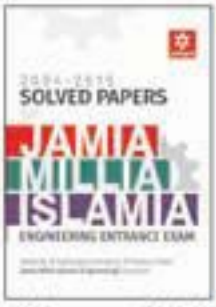

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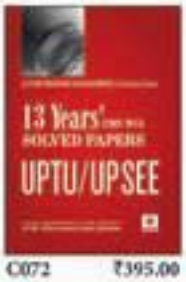

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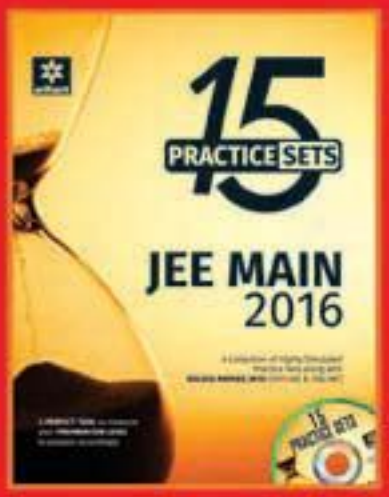

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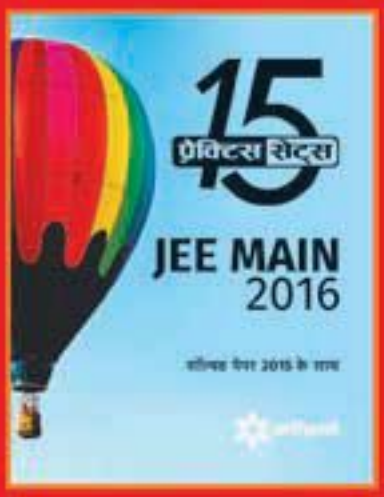

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